Blue Pelican Calculus

First Semester

Teacher Version 1.01

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Calculus AP Syllabus (First Semester)

Unit 1: Function limits and continuity

Lesson 01: Limit fundamentals, definitions

Lesson 02: Limits of rational and graphed functions

Lesson 03: Limit theorems, limits of trig functions

Lesson 04: Limits involving infinity

Lesson 05: Piecewise functions and continuity

Unit 1 review
Unit 1 test

Unit 2: Derivative fundamentals

Lesson 01: Average and instantaneous rates of change
Definition of the derivative at x = c

Lesson 02: Equations of tangent and normal lines

Lesson 03: Formal definition of the derivative

Lesson 04: A graphical look at derivatives

Lesson 05: Differentiability

Unit 2 review
Unit 2 test

Unit 3: Derivatives formulas; derivative of trig and piecewise functions

Lesson 01: Constant and power rules

Lesson 02: Product and quotient rules

Lesson 03: Trig function derivatives

Lesson 04: Linear approximations
Derivatives of piecewise functions

Lesson 05: Calculator derivatives

Cumulative review, unit 3
Unit 3 review
Unit 3 test

Unit 4: Chain Rule; higher order derivatives, applied rates of change

Lesson 01: Chain rule fundamentals

Lesson 02: Chain rule applied to trig functions

Lesson 03: Higher order derivatives

Lesson 04: Applied rates of change
Velocity, speed, and acceleration:

Cumulative review
Unit 4 review
Unit 4 test

Unit 5: Implicit differentiation

Lesson 01: Implicit differentiation fundamentals

Lesson 02: Tangent and normal lines (with implicit derivatives)
Implicit higher order derivatives

Lesson 03: Related rates

Lesson 04: More related rate problems

Cumulative review
Unit 5 review
Unit 5 test

Unit 6: Rolle's Theorem and the Mean Value Theorem
First and second derivative tests; Critical values

Lesson 1: Rolle's Theorem and the Mean Value Theorem

Lesson 2: First derivative test: Increasing/decreasing intervals
Critical values

Lesson 3: Local and absolute extrema

Lesson 4: Second derivative test: Concavity

Lesson 5: Graphs relating f(x), f '(x), and f ''(x)

Cumulative review
Unit 6 review
Unit 6 test

Unit 7: Optimization (maximizing & minimizing)

Lesson 1: Optimization problems

Lesson 2: More optimization problems

Lesson 3: Still more optimization problems

Cumulative review
Unit 7 test

Unit 8: Derivatives of inverse, exponential, and logarithm functions

Lesson 1: Fundamentals of inverse functions and their derivatives

Lesson 2: Derivatives of inverse trig functions

Lesson 3: Derivatives of exponential functions

Lesson 4: Derivatives of logarithm functions

Cumulative review
Unit 8 review
Unit 8 test

Unit 9: Antiderivatives (Indefinite integrals)

Lesson 1: Basic integration rules, integrating polynomials

Lesson 2. More integration practice

Lesson 3: Integrating trig functions

Lesson 4: Integration using the chain rule in reverse

Lesson 5: Applications of integration, evaluation of integration constants

Lesson 6: Indefinite integrals with a graphing calculator

Unit 9 review
Unit 9 test

Semester summary
    Semester review
    Semester test

Enrichment Topics

Topic A: Special sine and cosine limits

Topic B: Formal definition of continuity

Topic C: Verification of the power rule

Topic D: Verification of the product and quotient rules

Topic E: Verification of rules for derivative of sine and cosine functions

Topic F: Verification of the Chain Rule

Topic G: Verification of derivatives of exponential functions

Topic H: Verification of derivatives of logarithm functions

Topic I: Verification of derivatives of inverse trig functions

Topic J: An argument in support of the Fundamental Theorem of Calculus

Topic k: Why the absolute value for the integral of 1/x?

Topic L: Partial fractions

Calculus, Unit 1

Function limits and continuity

Unit 1: Lesson 01 Limit fundamentals, definitions

Consider

$$\lim_{x \to 2}(x^2 - 5x)$$

Read this as either

"The limit as *x* goes to two, of *x* squared minus five *x*."

or

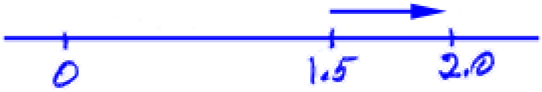
"The limit of *x* squared minus five *x*, **as *x* goes to two**."

The answer to the above limit can be thought of as the value that the function $y = f(x) = x^2 - 5x$ approaches as *x* gets closer and closer to 2.

> Let *f(x)* be a function defined at every number in an open interval containing *a*, except possibly at *a* itself. If the function values of *f(x)* approach a specific number *L* as *x* approaches *a*, then *L is the limit of f(x) as x approaches a*.

For the function $x^2 - 5x$ above, let *x* approach 2 in a table as follows (Consult **Calculator Appendix AE** and an associated video for how to produce this table on a graphing calculator):

x	$f(x) = x^2 - 5x$
1.5	**–5.25**
1.6	**–5.44**
1.7	**–5.61**
1.8	**–5.76**
1.9	**–5.89**
2.0	**–6.0**



In the table above, the right column (the function value) seems to approach –6 and, in fact, is exactly –6 at x = 2.

For the same function let's approach x = 2 **from the right** now instead of the left.

x	$f(x) = x^2 - 5x$
2.5	**–6.25**
2.4	**–6.24**
2.3	**–6.21**
2.2	**–6.16**
2.1	**–6.09**
2.0	**–6.0**

Again, the limit seems to be approaching –6. Notice that for our function $f(x) = x^2 – 5x$, $f(2) = 2^2 – 5(2) = –6$.

So why use the tables to find what the function value approaches as *x* approaches 2? **Why not just evaluate f(2) and be done with it?**

The fact is, we can do exactly that if the function is a **polynomial**.

If f(x) is a polynomial, then:

$$\lim_{x\to a} f(x) = f(a)$$

Example 1: Evaluate

$$\lim_{x\to 3}(2x^3 - x + 1)$$

$$f(3) = 2(3)^3 - 3 + 1 = 2(27) - 2 = 54 - 2 = \boxed{52}$$

Approaching from the left: Consider this table from the previous page. Notice that we are approaching 2 from the left. The notation for this **one sided limit** is:

$$\lim_{x\to 2^-} f(x) = -6$$

x	$f(x) = x^2 – 5x$
1.5	–5.25
1.6	–5.44
1.7	–5.61
1.8	–5.76
1.9	–5.89
2.0	–6.0

Approaching from the right: Consider this table from the previous page. Notice that we are approaching 2 from the right. The notation for this **one sided limit** is:

$$\lim_{x\to 2^+} f(x) = -6$$

x	$f(x) = x^2 – 5x$
2.5	–6.25
2.4	–6.24
2.3	–6.21
2.2	–6.16
2.1	–6.09
2.0	–6.0

Only when the limits of a function from **both left and right agree** can we say what the limit is in general:

$$\text{If} \quad \lim_{x \to a^-} f(x) = L \quad \text{and} \quad \lim_{x \to a^+} f(x) = L \quad \text{then} \quad \boldsymbol{\lim_{x \to a} f(x) = L}$$

In the following two examples, state the general limit in limit notation and the numeric answer (if it exists).

Example 2:

$\lim_{x \to 3^-} f(x) = 11$ and $\lim_{x \to 3^+} f(x) = 11$

$\lim_{x \to 3} F(x) = 11$

Example 3:

$\lim_{x \to 2^-} f(x) = -4$ and $\lim_{x \to 2^+} f(x) = 4$

$-4 \neq 4$

$\lim_{x \to 2} F(x) =$ D.N.E

(Does not exist)

Assignment:

1. Write out this limit expression in words.

$$\lim_{x\to -4}(x^3+1)$$

The limit as x goes to negative four of x cubed plus 1.

---

2. Convert "The limit of the square root of *x* plus 1, plus *x*, minus 3, as *x* goes to 17" into the mathematical notation for limits.

$$\lim_{x\to 17}(\sqrt{x+1}+x-3)$$

---

3. Evaluate

$\lim_{x\to -4}(x^2+8x-1)$

$$= f(-4) = (-4)^2 + 8(-4) - 1$$
$$= 16 - 32 - 1$$
$$= -16 - 1 = \boxed{-17}$$

4. Evaluate

$\lim_{x\to 1}(-5x^3+x^2+2)$

$$= f(1) = -5\cdot 1^3 + 1^2 + 2$$
$$= -5 + 1 + 2$$
$$= -5 + 3 = \boxed{-2}$$

In problems 5 – 8, state the problem in limit notation and what it seems to be approaching. If no apparent limit exists, then so state.

5.

x	f(x)
4.4	13.1
4.49	13.01
4.499	13.001
4.4999	13.0001
4.49999	13.00001

$$\lim_{x\to 4.5^-} f(x) = 13$$

6.

x	f(x)
−11.2	−.1
−11.18	−.09
−11.10	−.009
−11.02	−.004
−11.001	−.001

$$\lim_{x\to -11^-} f(x) = 0$$

7.

x	f(x)
4.4	13.1
4.49	13.2
4.499	13.4
4.4999	13.7
4.49999	14.1

No apparent limit

8.

x	f(x)
2.2	17
2.1	17
2.01	17
2.001	17
2.0001	17

$$\lim_{x\to 2^+} f(x) = 17$$

9. Write out this limit statement in words.

$$\lim_{x\to a^+}(x^3+1) = \mathrm{m}$$

The limit as x approaches a from the right, of x cubed plus one equals m.

10. Convert "The limit as x approaches b from the left, of f(x)." into mathematical terminology using limit notation.

$$\lim_{x\to b^-} f(x)$$

In problems 11-14, use the two one-sided limits to state the general limit in limit notation and the numeric answer (if it exists).

11.

$\lim_{x\to 0^-} \mathrm{f(x)} = -1$ and $\lim_{x\to 0^+} \mathrm{f(x)} = -1$

$$\lim_{x\to 0} f(x) = -1$$

12.

$\lim_{x\to 47^-} \mathrm{f(x)} = 0$ and $\lim_{x\to 47^+} \mathrm{f(x)} = 0$

$$\lim_{x\to 47} f(x) = 0$$

13. $f(x) = x^2 - x - 1$

$\lim_{x\to 0^-} f(x) = -1$ & $\lim_{x\to 0^+} f(x) = -1$

$\lim_{x\to 0} f(x) = f(0)$

$= 0^2 - 0 - 1$

$= \boxed{-1}$

14. $f(x) = 1/(x-5)$

$\lim_{x\to 5^-} f(x) = ?$ & $\lim_{x\to 5^+} f(x) = ?$

$\lim_{x\to 5^-} f(x) = -\infty$

$\lim_{x\to 5^+} f(x) = \infty$

They don't agree

No limit.

Unit 1:
Lesson 02

Limits of rational and graphed functions

To find $\lim\limits_{x \to a} f(x)$

- If f(x) is a polynomial, simply evaluate f(a).
- If f(x) is not a polynomial (such as a rational expression), try to evaluate f(a) unless it gives some indeterminate form such as:
  - Division by zero
  - Undefined
  - ∞/∞, 0/0, etc.

    In these cases, try to algebraically eliminate the difficulty before substituting in the *a* value.

Example 1: Find

$$\lim_{x \to 3}\left(\frac{x}{x+2}\right)$$

$$= \frac{3}{3+2} = \boxed{\frac{3}{5}}$$

Example 2: Find

$$\lim_{x \to 3}\left(\frac{x^2+2x-15}{x-3}\right)$$

$$= \lim_{x \to 3} \frac{(x+5)\cancel{(x-3)}}{\cancel{x-3}}$$

$$= \lim_{x \to 3} (x+5)$$

$$= 3+5 = \boxed{8}$$

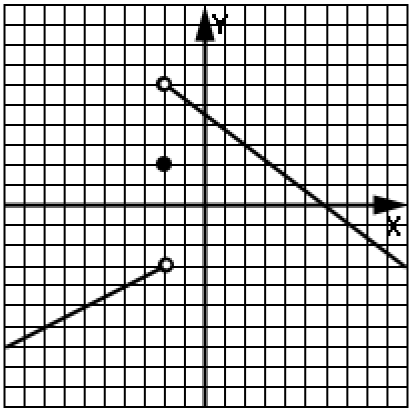
Example 3: Find

$$\lim_{x \to 4}\left(\frac{\sqrt{x}-2}{x-4}\right)$$

$$= \lim_{x \to 4} \frac{(\sqrt{x}-2)(\sqrt{x}+2)}{(x-4)(\sqrt{x}+2)} = \lim_{x \to 4} \frac{\cancel{x-4}}{\cancel{(x-4)}(\sqrt{x}+2)}$$

$$= \lim_{x \to 4} \frac{1}{\sqrt{x}+2} = \frac{1}{\sqrt{4}+2} = \frac{1}{2+2} = \boxed{\frac{1}{4}}$$

Example 4: For f(x) = y, find



$\lim_{x\to -2^-} f(x) = \mathbf{-3}$

← These two don't agree

$\lim_{x\to -2^+} f(x) = \mathbf{6}$

$\lim_{x\to -2} f(x) = \mathbf{D.N.E.\ (does\ not\ exist)}$

$f(-2) = \mathbf{2}$

Assignment: Find the indicated limits.

1. $\lim\limits_{x\to-2}\dfrac{x^2-5x-14}{x+2}$

$$=\lim_{x\to-2}\frac{(x-7)\cancel{(x+2)}}{\cancel{x+2}}$$

$$=\lim_{x\to-2}(x-7)=-2-7$$

$$=\boxed{-9}$$

2. $\lim\limits_{x\to6}(x^2+x-2)$

$$=6^2+6-2$$

$$=36+6-2$$

$$=\boxed{40}$$

3. $\lim\limits_{x\to5}\dfrac{\sqrt{x}-\sqrt{5}}{x-5}$

$$=\lim_{x\to5}\frac{\sqrt{x}-\sqrt{5}}{x-5}\cdot\frac{\sqrt{x}+\sqrt{5}}{\sqrt{x}+\sqrt{5}}$$

$$=\lim_{x\to5}\frac{\cancel{x-5}}{\cancel{(x-5)}(\sqrt{x}+\sqrt{5})}$$

$$=\lim_{x\to5}\frac{1}{\sqrt{x}+\sqrt{5}}=\frac{1}{\sqrt{5}+\sqrt{5}}$$

$$=\frac{1}{2\sqrt{5}}\cdot\frac{\sqrt{5}}{\sqrt{5}}=\frac{\sqrt{5}}{2\cdot5}=\boxed{\frac{\sqrt{5}}{10}}$$

4. $\lim\limits_{x\to1}\dfrac{x-1}{x^2-4x+3}$

$$=\lim_{x\to1}\frac{\cancel{x-1}}{\cancel{(x-1)}(x-3)}$$

$$=\lim_{x\to1}\frac{1}{x-3}$$

$$=\frac{1}{1-3}=\frac{1}{-2}$$

$$=\boxed{-\frac{1}{2}}$$

5. $\lim\limits_{x\to4}\dfrac{5x-20}{x^2-16}$

$$=\lim_{x\to4}\frac{5\cancel{(x-4)}}{\cancel{(x-4)}(x+4)}$$

$$=\lim_{x\to4}\frac{5}{x+4}$$

$$=\frac{5}{4+4}=\boxed{\frac{5}{8}}$$

6. $\lim\limits_{x\to4}\dfrac{\sqrt{x}-2}{x+4}$

$$=\frac{\sqrt{4}-2}{4+4}$$

$$=\frac{2-2}{8}=\frac{0}{8}$$

$$=\boxed{0}$$

7. $\lim_{x\to 2}\frac{x^4-16}{x^2-4}$

$$=\lim_{x\to 2}\frac{\cancel{(x^2-4)}(x^2+4)}{\cancel{x^2-4}}$$

$$=\lim_{x\to 2}(x^2+4)$$

$$=2^2+4=\boxed{8}$$

8. $\lim_{x\to 9}\frac{\sqrt{x}-3}{x-9}$

$$=\lim_{x\to 9}\frac{\sqrt{x}-3}{x-9}\;\frac{\sqrt{x}+3}{\sqrt{x}+3}$$

$$=\lim_{x\to 9}\frac{\cancel{x-9}}{\cancel{(x-9)}(\sqrt{x}+3)}$$

$$=\lim_{x\to 9}\frac{1}{\sqrt{x}+3}$$

$$=\frac{1}{\sqrt{9}+3}=\frac{1}{3+3}=\boxed{\frac{1}{6}}$$

9. $\lim_{x\to 0}(x^2-5x-1)$

$$=0^2-5\cdot 0-1$$

$$=\boxed{-1}$$

10. $\lim_{x\to 2}\frac{|x+2|}{x+2}$

$$=\frac{|2+2|}{2+2}=\frac{|4|}{4}$$

$$=\frac{4}{4}=\boxed{1}$$

11. $\lim_{x\to 5}\frac{5-x}{\sqrt{x}-\sqrt{5}}$

$$=\lim_{x\to 5}\frac{5-x}{\sqrt{x}-\sqrt{5}}\;\frac{\sqrt{x}+\sqrt{5}}{\sqrt{x}+\sqrt{5}}$$

$$=\lim_{x\to 5}\frac{(5-x)(\sqrt{x}+\sqrt{5})}{(x-5)}\;\frac{-1}{-1}$$

$$=\lim_{x\to 5}\frac{-\cancel{(5-x)}(\sqrt{x}+\sqrt{5})}{\cancel{(5-x)}}$$

$$=-(\sqrt{5}+\sqrt{5})=\boxed{-2\sqrt{5}}$$

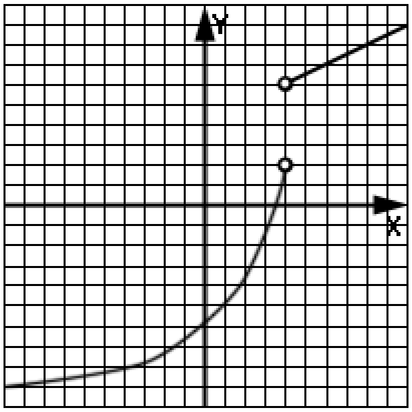
12. $\lim_{x\to 9}\frac{9-x}{81-x^2}$

$$=\lim_{x\to 9}\frac{\cancel{9-x}}{\cancel{(9-x)}(9+x)}$$

$$=\lim_{x\to 9}\frac{1}{9+x}$$

$$=\frac{1}{9+9}=\boxed{\frac{1}{18}}$$

13. Find the following for f(x) = y



$\lim_{x\to 4^-} f(x) = \mathbf{2}$

$\lim_{x\to 4^+} f(x) = \mathbf{6}$

$\lim_{x\to 4} f(x) = \textbf{does not exist}$

$\lim_{x\to 6} f(x) = \mathbf{7}$

$f(4) = \textbf{does not exist}$

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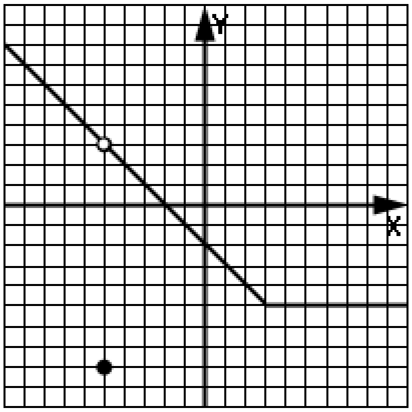
14. Find the following for f(x) = y



$\lim_{x\to -5^-} f(x) = \mathbf{3}$

$\lim_{x\to -5^+} f(x) = \mathbf{3}$

$\lim_{x\to -5} f(x) = \mathbf{3}$

$\lim_{x\to 3} f(x) = \mathbf{-5}$

$f(-5) = \mathbf{-8}$

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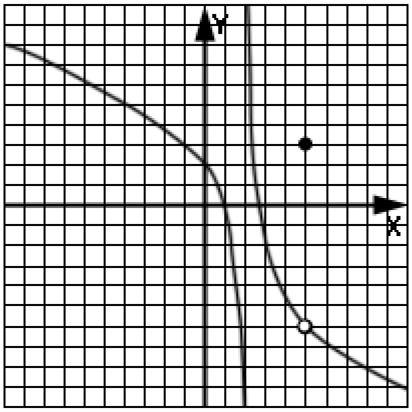
15. Find the following for f(x) = y



$\lim_{x\to 2^-} f(x) = -\infty \textbf{ or D. N. E.}$

$\lim_{x\to 2^+} f(x) = +\infty \textbf{ or D. N. E}$

$\lim_{x\to 5} f(x) = \mathbf{-6}$

$f(2) = \textbf{D. N. E.}$

$f(5) = \mathbf{3}$

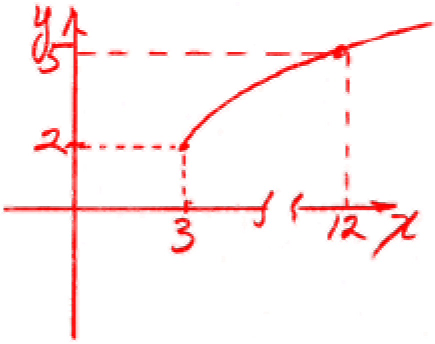
16. Sketch the function, $f(x) = \sqrt{x-3} + 2$. Use the sketch to find the following limits.

$\lim_{x \to 3^-} f(x) = \mathbf{D.N.E.}$

$\lim_{x \to 3^+} f(x) = \mathbf{2}$

$\lim_{x \to 3} f(x) = \mathbf{D.N.E.}$

$\lim_{x \to 12} f(x) = \mathbf{5}$



Unit 1: Lesson 03

Limit theorems, limits of trig functions

For the following limit theorems, assume:

$$\lim_{x\to a} f(x) = L \quad \text{and} \quad \lim_{x\to a} g(x) = M$$

$$\lim_{x\to a}[f(x) \pm g(x)] = \lim_{x\to a} f(x) \pm \lim_{x\to a} g(x) = L \pm M$$

$$\lim_{x\to a}[f(x)g(x)] = \left[\lim_{x\to a} f(x)\right] \cdot \left[\lim_{x\to a} g(x)\right] = L \cdot M$$

$$\lim_{x\to a} \frac{f(x)}{g(x)} = \frac{\lim_{x\to a} f(x)}{\lim_{x\to a} g(x)} = \frac{L}{M}$$

$$\lim_{x\to a}(k \cdot f(x)) = k \lim_{x\to a} f(x) = k L$$

(where k is a constant)

Example 1: Assume $\lim_{x\to 3} f(x) = -1$ and $\lim_{x\to 3} g(x) = 7$

$$\lim_{x\to 3}[3g(x) - f(x)] =?$$

$$= \lim_{x\to 3} 3g(x) - \lim_{x\to 3} f(x) = 3\overset{7}{\cancel{\lim_{x\to 3} g(x)}} - \overset{-1}{\cancel{\lim_{x\to 3} f(x)}}$$

$$= 3 \cdot 7 + 1 = 21 + 1 = \boxed{22}$$

$$\lim_{x\to 3} \frac{x + f(x)}{g(x) - f(x)}$$

$$= \lim_{x\to 3}[x + f(x)] \Big/ \lim_{x\to 3}[g(x) - f(x)]$$

$$= \left[3 + \lim_{x\to 3} f(x)\right] \Big/ \left[\lim_{x\to 3} g(x) - \lim_{x\to 3} f(x)\right]$$

$$= [3 + (-1)] / [7 - (-1)] = 2/8 = \boxed{\tfrac{1}{4}}$$

The following special trig limits should be memorized (See **Enrichment Topic A** for their justification):

$$\lim_{x\to0}\frac{\sin(x)}{x} = \lim_{x\to0}\frac{x}{\sin(x)} = 1$$

$$\lim_{x\to0}\frac{1-\cos(x)}{x} = 0$$

The following trig approximations are useful as ***x* (in radians) approaches 0**.

$\sin(x) \approx x$ … comes from $\sin(x) = x - x^3/3! + x^5/5! - x^7/7! + \ldots$

$\cos(x) \approx 1$ … comes from $\cos(x) = 1 - x^2/2! + x^4/4! - x^6/6! + \ldots$

In finding the limits of trig functions, use direct substitution first. If that yields an indeterminate form, then use one of the special cases above.

Example 2: $\lim_{\theta\to0}\frac{\sin(8\theta)}{\theta} = ?$

$$= \lim_{\theta\to0}\frac{8\sin(8\theta)}{8\theta} = \lim_{\theta\to0} 8\underbrace{\frac{\sin(8\theta)}{8\theta}}_{1}$$

$$= \boxed{8}$$

Example 3: $\lim_{\alpha\to0}\tan(\alpha) = ?$

Use direct substitution:

$= \tan(0) = \boxed{0}$

Assignment: For problems 1- 4, assume the following:

$$\lim_{x\to -4} f(x) = 1 \quad \text{and} \quad \lim_{x\to -4} g(x) = -2$$

1. $\lim_{x\to -4} \frac{g(x)}{f(x)+x}$

$= \frac{\lim_{x\to -4} g(x)}{\lim_{x\to -4}[f(x)+x]}$

$= \frac{-2}{1+(-4)}$

$= \frac{-2}{-3} = \boxed{\frac{2}{3}}$

2. $\lim_{x\to -4}[xf(x) - g(x)]$

$= \lim_{x\to -4} x f(x) - \lim_{x\to -4} g(x)$

$= -4(1) - (-2)$

$= -4+2 = \boxed{-2}$

3. $\lim_{x\to -4}[f(x)^2 - g(x)^2 - 2]$

$= \left[\lim_{x\to -4} f(x)\right]^2 - \left[\lim_{x\to -4} g(x)\right]^2 - 2$

$= 1^2 - (-2)^2 - 2$

$= 1 - 4 - 2$

$= \boxed{-5}$

4. $\lim_{x\to -4}[f(x) + g(x)]^2$

$= \left[\lim_{x\to -4} f(x) + \lim_{x\to -4} g(x)\right]^2$

$= [1 + (-2)]^2$

$= [-1]^2 = \boxed{1}$

5. $\lim_{x\to 0} \frac{\tan(x)}{x} = ?$

$= \lim_{x\to 0} \frac{\sin(x)}{\cos(x)\,x} = \lim_{x\to 0} \frac{\sin(x)}{x}\frac{1}{\cos(x)}$

$= \underset{1}{\cancel{\left[\lim_{x\to 0} \frac{\sin x}{x}\right]}} \left[\lim_{x\to 0} \frac{1}{\cos x}\right] = 1 \, \frac{1}{\cos 0} = \frac{1}{1} = \boxed{1}$

6. $\lim\limits_{x\to 0} \dfrac{-\sin(\pi\, x)}{\pi\, x} = ?$

$$\lim_{x\to c}\left[-1\,\frac{\sin(\pi x)}{\pi x}\right] = -1\left[\cancelto{1}{\lim_{x\to 0}\frac{\sin(\pi x)}{\pi x}}\right]$$

$$= \boxed{-1}$$

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7. $\lim\limits_{\theta\to 0} \dfrac{\cos^2(\theta) - 1}{\theta} = ?$

$$= \lim_{\theta\to 0}\frac{(\cos\theta-1)(\cos\theta+1)}{\theta} = \cancelto{0}{\lim_{\theta\to 0}\frac{-1(1-\cos\theta)}{\theta}}\ \lim_{\theta\to 0}(\cos\theta+1)$$

$$= 0(\cos 0+1) = 0(1+1) = 0(2) = \boxed{0}$$

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8. $\lim\limits_{x\to \pi} \cos(x) = ?$

direct substitution

$= \cos(\pi) = \boxed{-1}$

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9. $\lim\limits_{b\to 0} \dfrac{(1-\cos(b))^2}{b} = ?$

$$= \lim_{b\to 0}\frac{(1-\cos b)(1-\cos b)}{b} = \left[\cancelto{0}{\lim_{b\to 0}\frac{1-\cos b}{b}}\right]\left[\lim_{b\to 0}\frac{1-\cos b}{1}\right]$$

$$= 0\left[1-\cos 0\right] = 0\left[1-1\right] = \boxed{0}$$

10. $\lim_{x\to 0} \frac{x}{\sin(7x)} = ?$

$= \lim_{x\to 0} \frac{7x}{7\sin(7x)} = \frac{1}{7} \lim_{x\to 0} \frac{7x}{\sin(7x)} \to 1$

$= \frac{1}{7}(1) = \boxed{\frac{1}{7}}$

---

11. $\lim_{\theta\to \pi/2} \frac{\cot(\theta)}{\cos(\theta)} = ?$

$= \lim_{\theta\to \pi/2} \frac{\cancel{\cos\theta}}{\sin\theta\ \cancel{\cos\theta}} = \lim_{\theta\to \pi/2} \frac{1}{\sin\theta}$

direct sub $\rightarrow$ $= \frac{1}{\sin(\pi/2)} = \frac{1}{1} = \boxed{1}$

---

12. $\lim_{x\to 5} \tan\left(\frac{\pi x}{4}\right) = ?$

direct subsitution

$= \tan\left(\frac{5\pi}{4}\right) = \boxed{+1}$

---

13. $\lim_{\beta\to 0} \frac{\sin(\beta)\cos(\beta)}{\beta}$

$\lim_{\beta\to 0} \frac{2\sin\beta\cos\beta}{2\beta}$

use $\sin 2\theta = 2\sin\theta\cos\theta$

$= \lim_{\beta\to 0} \frac{\sin(2\beta)}{2\beta} \to 1$ $= \boxed{1}$

14. $\lim_{x \to 9} \frac{18 - 2x}{3 - \sqrt{x}} = ?$

$$= \lim_{x \to 9} \frac{2(9-x)}{3-\sqrt{x}} \cdot \frac{3+\sqrt{x}}{3+\sqrt{x}} = \lim_{x \to 9} \frac{2\cancel{(9-x)}(3+\sqrt{x})}{\cancel{9-x}}$$

$$= \lim_{x \to 9} \frac{2(3+\sqrt{x})}{1} = 2(3+\sqrt{9})$$

$$= 2(3+3)$$

$$= \boxed{12}$$

Unit 1: Lesson 04 Limits involving infinity

A fundamental limit on which many others depend is:

$$\lim_{x\to\infty}\frac{1}{x^n}=0 \quad ;\text{n is any positive power}$$

x	f(x) = 1/x^1
100	.01
1,000	.001
10,000	.0001
100,000	.00001
1,000,000	.000001

Infinity (∞) is **not a position on the number line.** Rather, it is a **concept** of a number continuing to get larger and larger without any limit. With that in mind, consider the problem:

$$\lim_{x\to\infty}\frac{3x^2}{x^2+5}$$

What happens if we try to “substitute in ∞” (which is illegal since ∞ is not a number)? We would illegally obtain the following:

$$\frac{3\infty^2}{\infty^2+5}=\frac{\infty}{\infty}$$

Can we just cancel ∞/∞ to make 1? No, because ∞ is not a number that could be canceled as could be done with 5/5. Example 1 below shows the proper way to handle this problem where the answer will be shown to be 3, not 1.

Example 1: $\lim_{x\to\infty}\frac{3x^2}{x^2+5}=?$

$$\lim_{x\to\infty}\frac{3x^2}{x^2+5}\cdot\frac{1/x^2}{1/x^2}=\lim_{x\to\infty}\frac{\frac{3x^2}{x^2}}{\frac{x^2}{x^2}+\frac{5}{x^2}}$$

$$=\lim_{x\to\infty}\frac{3}{1+5/x^2}=\frac{3}{1+0}=\boxed{3}$$

As a general rule in handling a problem such as Example 1, find the **highest degree** in both the numerator and denominator (assume it's *n*) and multiply by 1 in this form:

$$\frac{\frac{1}{x^n}}{\frac{1}{x^n}}$$

Example 2: $\lim\limits_{x\to\infty} \frac{7x^2 - 2x}{4x^3 - x} = ?$

$$\lim_{x\to\infty} \frac{7x^2-2x}{4x^3-x} \cdot \frac{\frac{1}{x^3}}{\frac{1}{x^3}} = \lim_{x\to\infty} \frac{\frac{7x^2}{x^3} - \frac{2x}{x^3}}{\frac{4x^3}{x^3} - \frac{x}{x^3}}$$

$$= \lim_{x\to\infty} \frac{\frac{7}{x} - \frac{2}{x^2}}{4 - \frac{1}{x^2}} = \frac{0-0}{4-0} = \frac{0}{4} = \boxed{0}$$

Example 3: $\lim\limits_{x\to\infty} (x^3 - 6x^2 + x) = ?$

Factor out x^3

$$\lim_{x\to\infty} \left[x^3\left(1 - \frac{6}{x} + \frac{1}{x^2}\right)\right]$$

$$= \left[\lim_{x\to\infty} x^3\right]\left[\lim_{x\to\infty}\left(1 - \frac{6}{x} + \frac{1}{x^2}\right)\right]$$

$$= (+\infty)[1 - 0 + 0] = +\infty(1) = \boxed{+\infty}$$

Notice in Example 3 that the other terms pale in comparison to x^3 as x goes to infinity. Therefore, we have the following rule:

For any polynomial, P(x)

$$\lim_{x\to\pm\infty} P(x) = \lim_{x\to\pm\infty} (\text{highest power term of } P(x))$$

Example 4: $\lim\limits_{x\to-\infty}(11x^2-2x^3+x)=?$

Factor out x^3

$$=\lim_{x\to-\infty}\left[x^3\left(\frac{11}{x}-2+\frac{1}{x^2}\right)\right]$$

$$=\left[\lim_{x\to-\infty}(x^3)\right]\left[\lim_{x\to-\infty}\left(\frac{11}{x}-2+\frac{1}{x^2}\right)\right]$$

$$=(-\infty)\left[0-2+0\right]=(-\infty)(-2)=\boxed{+\infty}$$

Consider the problem $\lim\limits_{x\to2^-}\dfrac{2x^2-5x+1}{x^2+x-6}$

Direct substitution of x = 2 yields: $\dfrac{-1}{0}$

So is the answer +∞ or –∞? Example 5 shows the correct way to analyze this problem.

Example 5: $\lim\limits_{x\to2^-}\dfrac{2x^2-5x+1}{x^2+x-6}=?$

$$=\lim_{x\to2^-}\frac{2x^2-5x+1}{(x-2)(x+3)}=\lim_{x\to2^-}\left(\frac{1}{x-2}\right)\left[\lim_{x\to2^-}\frac{2x^2-5x+1}{x+3}\right]$$

$$=[-\infty]\left[(2(2)^2-5(2)+1)/(2+3)\right]$$

$$=[-\infty]\ \frac{[8-10+1]}{5}=[-\infty]\ [-1/5]$$

$$=\boxed{+\infty}$$

Assignment:

1. $\lim\limits_{x\to\infty} \dfrac{x+5}{x-2} = ?$

$$\lim_{x\to\infty} \frac{x+5}{x-2} \; \frac{\frac{1}{x}}{\frac{1}{x}} = \lim_{x\to\infty} \frac{\frac{x}{x}+\frac{5}{x}}{\frac{x}{x}-\frac{2}{x}}$$

$$= \lim_{x\to\infty} \frac{1+\frac{5}{x}}{1-\frac{2}{x}} = \frac{1+0}{1-0} = \frac{1}{1} = \boxed{1}$$

---

2. $\lim\limits_{x\to\infty} \dfrac{5x^3+2}{20x^3-6x}$

$$\lim_{x\to\infty} \frac{5x^3+2}{20x^3-6x} \; \frac{\frac{1}{x^3}}{\frac{1}{x^3}} = \lim_{x\to\infty} \frac{\frac{5x^3}{x^3}+\frac{2}{x^3}}{\frac{20x^3}{x^3}-\frac{6x}{x^3}}$$

$$= \lim_{x\to\infty} \frac{5+\frac{2}{x^3}}{20-\frac{6}{x^2}} = \frac{5+0}{20-0} = \boxed{\frac{1}{4}}$$

---

3. $\lim\limits_{x\to\infty} \dfrac{5+2^x}{15-6x}$

$$\lim_{x\to\infty} \frac{5+2^x}{15-6x} \; \frac{\frac{1}{2^x}}{\frac{1}{2^x}} = \lim_{x\to\infty} \frac{\frac{5}{2^x}+\cancel{\frac{2^x}{2^x}}}{\frac{15}{2^x}-\frac{6x}{2^x}} = \lim_{x\to\infty} \frac{\frac{5}{2^x}+1}{\frac{15-6x}{2^x}}$$

pos

neg for this term

$$= \frac{(0+1)^+}{0^-} = \boxed{-\infty}$$

---

4. $\lim\limits_{x\to\infty} (7-11x^2-6x^5)$

$$= \lim_{x\to\infty} (-6x^5) = -6 \lim_{x\to\infty} x^5 = -6(\infty^5) \to +\infty$$

$$= \boxed{-\infty}$$

5. $\lim\limits_{x\to\infty} 6^x = ?$

$$= 6^{\infty} = \boxed{\infty}$$

---

6. $\lim\limits_{x\to\infty} \dfrac{9x^4 - x^3 + 1}{x - 2x^4}$

$$\lim_{x\to\infty} \frac{9x^4 - x^3 + 1}{x - 2x^4} \cdot \frac{\frac{1}{x^4}}{\frac{1}{x^4}} = \lim_{x\to\infty} \frac{\frac{9x^4}{x^4} - \frac{x^3}{x^4} + \frac{1}{x^4}}{\frac{x}{x^4} - \frac{2x^4}{x^4}}$$

$$= \lim_{x\to\infty} \frac{9 - \frac{1}{x} + \frac{1}{x^4}}{\frac{1}{x^3} - 2} = \frac{9 - 0 + 0}{0 - 2} = \boxed{\frac{-9}{2}}$$

---

7. $\lim\limits_{x\to-\infty} (12x^4 - x^3 + 7x^2 + 1)$

$= \lim\limits_{x\to-\infty} (12x^4)$ highest power term → $12x^4$

$$= 12(-\infty)^4 = \boxed{\infty}$$

---

8. $\lim\limits_{x\to\infty} \left(7 - \dfrac{1}{x} + \dfrac{1}{x^2}\right)$

$$= 7 - 0 + 0 = \boxed{7}$$

9. $$\lim_{x\to1^+}\frac{3x^4-x+1}{x^2-6x+5}$$

$$\lim_{x\to1^+}\frac{3x^4-x+1}{x^2-6x+5}=\lim_{x\to1^+}\frac{3x^4-x+1}{(x-1)(x-5)}$$

$$=\lim_{x\to1^+}\frac{1}{x-1}\,\frac{3x^4-x+1}{x-5}=\left[\lim_{x\to1^+}\frac{1}{x-1}\right]\left[\frac{3\cdot1\cancel{-1+1}}{1-5}\right]=\frac{\infty}{-4}=\boxed{-\infty}$$

$+\infty$

---

10. $$\lim_{x\to-\infty}(x^3-2{,}000{,}000)$$

$$=\lim_{x\to-\infty}(x^3)=(-\infty)^3=\boxed{-\infty}$$

---

11. $$\lim_{x\to1^-}\frac{x^2-2x+1}{x^3-3x^2+3x-1}$$

Direct substitution yields 0/0, so we know (x-1) is a factor

$$\begin{array}{r} x^2-2x+1 \\ x-1\overline{)x^3-3x^2+3x-1} \\ \underline{\cancel{x^3}-x^2} \\ \cancel{-2x^2}+3x \\ \underline{-2x^2+2x} \\ x-1 \end{array}$$

$$\lim_{x\to1^-}\frac{\cancel{x^2-2x+1}}{(x-1)\cancel{(x^2-2x+1)}}$$

$$=\lim_{x\to1^-}\frac{1}{(x-1)}=\boxed{-\infty}$$

---

12. $$\lim_{x\to4^-}\frac{x^{50}-3x^{49}}{x-4}$$

$$\lim_{x\to4^-}\frac{x^{49}(x-3)}{x-4}=\left[\lim_{x\to4^-}x^{49}\right]\left[\lim_{x\to4^-}\frac{x-3}{x-4}\right]$$

$$=[\infty]\left[\lim_{x\to4^-}\frac{x-3}{(x-4)}\right]=[\infty]\left[\frac{1}{-\underset{0}{\cancel{\#}}}\right]=\infty(-\infty)$$

$$=\boxed{-\infty}$$

Unit 1: Lesson 05

Piecewise functions and continuity

A function is discontinuous at a particular *x* value if we need to "**lift the pencil**" at that point in order to keep drawing that function. Otherwise, the function is said to be continuous there. See **Enrichment Topic B** for a more formal definition of discontinuity.

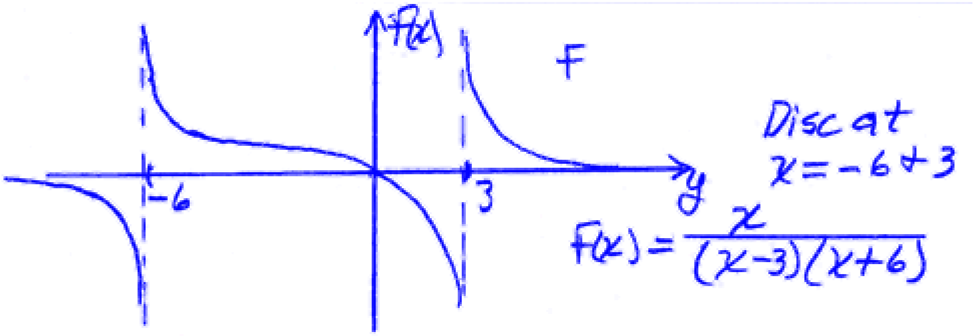
There are several things that can cause a discontinuity at x = a for a function:

- There is a vertical asymptote at x = a . Typically, (x – a) is a factor of the denominator. (See Example 1).
- A piecewise function abruptly "jumps" at x = a. (See Example 3.)
- There is a "hole" in the graph at x = a. (See Example 5.)

Polynomials are continuous everywhere.

Example 1: Sketch the graph of f(x) (and note the positions of any discontinuities).

$$f(x) = \frac{x}{x^2 + 3x - 18}$$



Example 2: Just by observing the sketch in Example 1, determine the following limits:

$\lim_{x \to -6^-} f(x) = -\infty$

$\lim_{x \to -6^+} f(x) = +\infty$

$\lim_{x \to 3^-} f(x) = -\infty$

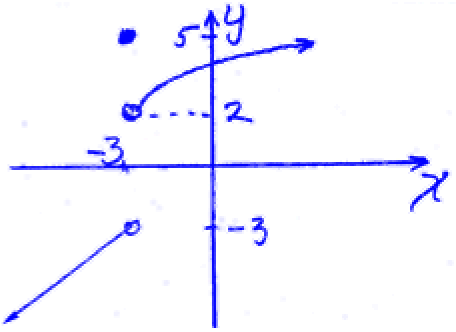
$\lim_{x \to 3^+} f(x) = +\infty$

$\lim_{x \to 3} f(x) = \textbf{D.N.E.}$

$f(3) = \textbf{D.N.E.}$

Example 3: Sketch this piecewise function.

$$f(x) = \begin{cases} x & when\ x < -3 \\ 5 & when\ x = -3 \\ \sqrt{x+3}+2 & when\ x > -3 \end{cases}$$



Example 4: Just by observing the sketch in Example 3, determine the following values:

$\lim_{x \to -3^-} f(x) = \mathbf{-3}$

$\lim_{x \to -3^+} f(x) = \mathbf{2}$

$\lim_{x \to -3} f(x) = \mathbf{D.N.E.}$

$f(-3) = \mathbf{5}$

Example 5: State the *x* positions of discontinuity and identify which are "holes."

x = –4 (hole) and x = 3

$\lim_{x \to -4^-} f(x) = \mathbf{1}$

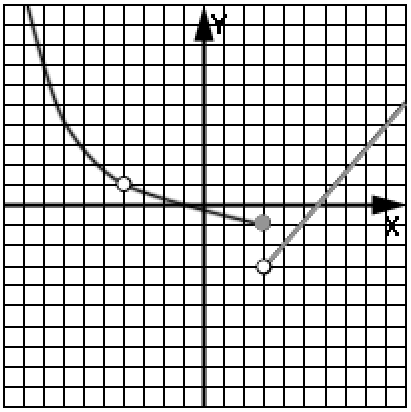
$\lim_{x \to -4^+} f(x) = \mathbf{1}$

$\lim_{x \to -4} f(x) = \mathbf{1}$

$\lim_{x \to 3^-} f(x) = \mathbf{-1}$

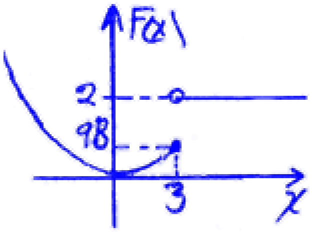
$\lim_{x \to 3^+} f(x) = \mathbf{-3}$

$\lim_{x \to 3} f(x) = \mathbf{D.N.E.}$



Example 6: Determine the value of B so as to insure that this function is everywhere continuous.

$$f(x) = \begin{cases} Bx^2 & if\ x \le 3 \\ 2 & if\ x > 3 \end{cases}$$



$f(3) = B\,3^2$

$= 9B$

$9B = 2$

$B = \boxed{\frac{2}{9}}$

Assignment: In problems 1-3, sketch the function and identify any positions of discontinuity.

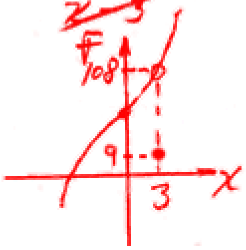
1. $$f(x) = \begin{cases} \dfrac{x^4-81}{x-3} & \text{if } x \neq 3 \\ 9 & \text{if } x = 3 \end{cases}$$

$$\frac{x^4-81}{x-3} = \frac{(x^2-9)(x^2+9)}{x-3}$$
$$= \frac{\cancel{(x-3)}(x+3)(x^2+9)}{\cancel{x-3}}$$

$\lim_{x\to 3^-} f(x) = \mathbf{108}$

$\lim_{x\to 3^+} f(x) = \mathbf{108}$

$\lim_{x\to 3} f(x) = \mathbf{108}$



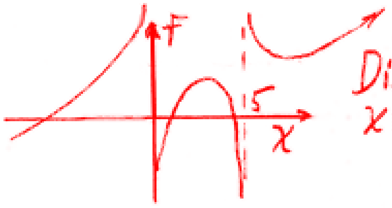
Disc at $x=3$

---

2. $$f(x) = 4 - \frac{1}{x} + \frac{x^2}{x-5}$$

$\lim_{x\to 0^-} f(x) = +\infty$

$\lim_{x\to 5^+} f(x) = +\infty$



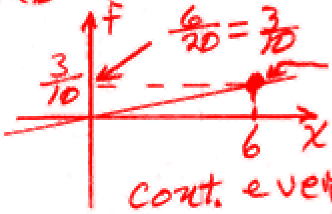
Disc at $x=0$ & $x=5$

---

3. $$f(x) = \begin{cases} \dfrac{x^2-6x}{20x-120} & \text{if } x \neq 6 \\ \dfrac{3}{10} & \text{if } x = 6 \end{cases}$$

$$\frac{x\cancel{(x-6)}}{20\cancel{(x-6)}} = \frac{x}{20}$$

$\frac{6}{20} = \frac{3}{10}$ The "hole" gets plugged



Cont. everywhere

$\lim_{x\to 6^-} f(x) = \mathbf{3/10}$

$\lim_{x\to 6^+} f(x) = \mathbf{3/10}$

$\lim_{x\to 6} f(x) = \mathbf{3/10}$

---

4. Algebraically "design" a linear function that has a hole at x = 2, but whose limit as x approaches 2 is 5.

$$f(x) = \frac{mx\cancel{(x-2)}}{\cancel{x-2}} = mx$$

$f(2) = 5$
$m2 = 5$
$m = \frac{5}{2}$

$$\boxed{f(x) = \frac{5x(x-2)}{2(x-2)}}$$

In problems 5-7, state the *x* positions of discontinuity and answer the questions.

5.

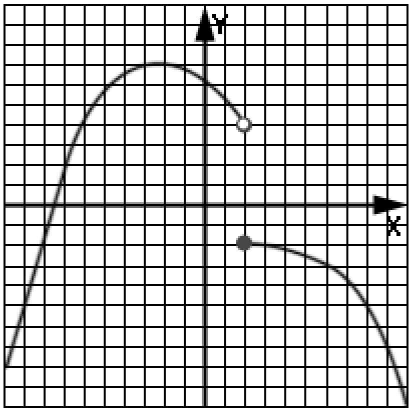
$\lim_{x \to 2^-} f(x) = \mathbf{4}$

$\lim_{x \to 2^+} f(x) = \mathbf{-2}$

$\lim_{x \to 2} f(x) = \mathbf{D.N.E.}$

$f(2) = \mathbf{-2}$

Disc at $x=2$



6.

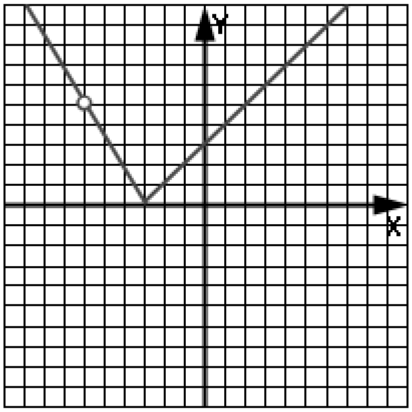
$\lim_{x \to -6^-} f(x) = \mathbf{5}$

$\lim_{x \to -6^+} f(x) = \mathbf{5}$

$\lim_{x \to -6} f(x) = \mathbf{5}$

$f(-6) = \mathbf{D.N.E.}$

Disc. at $x=-6$



7.

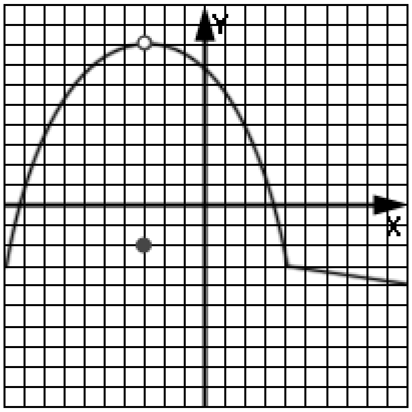
$\lim_{x \to -3^-} f(x) = \mathbf{8}$

$\lim_{x \to -3^+} f(x) = \mathbf{8}$

$\lim_{x \to -3} f(x) = \mathbf{8}$

$f(-3) = \mathbf{-2}$

Disc. at $x=-3$



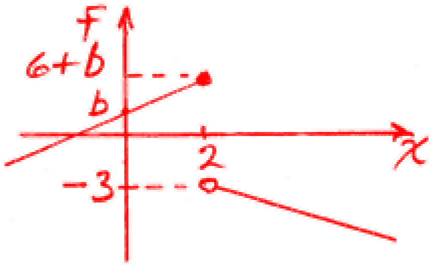
---

8. State the position of discontinuity of $f(x) = 8x^4 - 3x^3 + x^2 - 6$

Cont. everywhere
It's a polynomial

9. Determine the value of *b* so as to insure that the function is everywhere continuous.

$$f(x) = \begin{Bmatrix} 3x + b & \text{if } x \le 2 \\ -x - 1 & \text{if } x > 2 \end{Bmatrix}$$

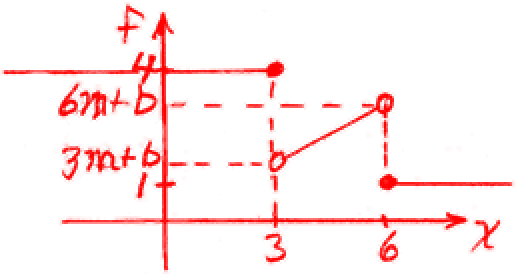


$$6 + b = -3$$
$$b = -3 - 6 = \boxed{-9}$$

---

10. Determine the values of *m* and *b* so as to insure that the function is everywhere continuous.

$$f(x) = \begin{Bmatrix} 4 & \text{if } x \le 3 \\ mx + b & \text{if } 3 < x < 6 \\ 1 & \text{if } x \ge 6 \end{Bmatrix}$$



$$\underbrace{3m + b = 4 \qquad 6m + b = 1}_{\text{solve simultaneously}}$$

$$3m + b = 4 \longrightarrow 3m + \cancel{b} = 4$$
$$-(6m + b) = 1(-1) \longrightarrow -6m - \cancel{b} = -1$$
$$-3m = 3$$
$$m = \boxed{-1}$$

$$3m + b = 4$$
$$3(-1) + b = 4$$
$$b = 4 + 3 = \boxed{7}$$

Unit 1:
Review

1. Write out this limit expression in words:

$$\lim_{x \to -5^+} f(x)$$

"The limit of f of *x* as *x* goes to negative 5 from the right."

In problems 2 and 3, state the problem in (one-sided) limit notation and what it seems to be approaching. If no apparent limit exists, then so state.

2.

x	f(x)
5.75	500
5.71	1002
5.7001	100,005
5.70002	2,000,500
5.700009	120,010,075

$$\lim_{x \to 5.7^+} f(x) = +\infty$$

3.

x	f(x)
−6.12	π/3
−6.11	π/100
−6.103	π/1000
−6.10054	π/100,000
−6.100003	π/1,000,000

$$\lim_{x \to -6.1^-} f(x) = 0$$

In problems 4 and 5, give the general limit (if it exists).

4. f(x) = x^2 −4x −1

$$\lim_{x \to 2} f(x) = ?$$

$$F(2) = 2^2 - 4(2) - 1$$
$$= 4 - 8 - 1 = \boxed{-5}$$

5. f(x) = 1/(x + 8)

$$\lim_{x \to -8} f(x) = ?$$

$$\left.\begin{aligned} \lim_{x \to -8^-} f(x) &= -\infty \\ \lim_{x \to -8^+} f(x) &= +\infty \end{aligned}\right]$$

Don't agree

$$\lim_{x \to -8} f(x) = D.N.E.$$

---

6. $$\lim_{x \to 6} \frac{x^2 + 4x - 12}{x + 6} = ?$$

$$\lim_{x \to 6} \frac{\cancel{(x+6)}(x-2)}{\cancel{x+6}} = \lim_{x \to 6} (x-2) = 6 - 2$$
$$= \boxed{4}$$

7. $\lim\limits_{x\to7}\dfrac{\sqrt{x}-\sqrt{7}}{x-7}=?$

$$\lim_{x\to7}\frac{\sqrt{x}-\sqrt{7}}{x-7}\cdot\frac{\sqrt{x}+\sqrt{7}}{\sqrt{x}+\sqrt{7}}=\lim_{x\to7}\frac{\cancel{(x-7)}}{\cancel{(x-7)}(\sqrt{x}+\sqrt{7})}$$

$$=\lim_{x\to7}\frac{1}{\sqrt{x}+\sqrt{7}}=\frac{1}{\sqrt{7}+\sqrt{7}}=\frac{1}{2\sqrt{7}}\cdot\frac{\sqrt{7}}{\sqrt{7}}=\frac{\sqrt{7}}{2\cdot7}$$

$$=\boxed{\frac{\sqrt{7}}{14}}$$

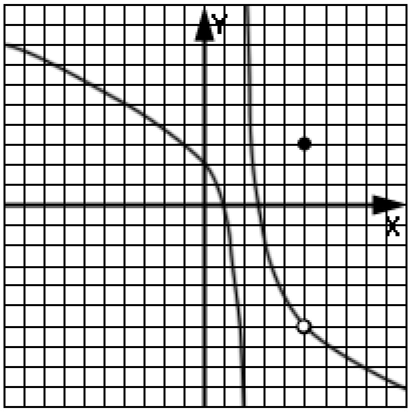
---

8. $\lim\limits_{x\to8}\dfrac{x}{|x+8|}$

use direct substitution $=\dfrac{8}{|8+8|}=\dfrac{8}{16}=\boxed{\dfrac{1}{2}}$

---

9. Find the following for f(x) = y



$\lim\limits_{x\to2^-}f(x)=-\infty$ **or D.N.E.**

$\lim\limits_{x\to2^+}f(x)=+\infty$ **or D.N.E**

$\lim\limits_{x\to5}f(x)=-6$

$f(2)=$ **D.N.E.**

$f(5)=3$

---

10. Assume $\lim\limits_{x\to-2}f(x)=-6$ and $\lim\limits_{x\to-2}g(x)=7$

$\lim\limits_{x\to-2}\dfrac{x+f(x)}{g(x)-f(x)}=?$

$$=\frac{-2+\lim_{x\to-2}f(x)}{\lim_{x\to-2}g(x)-\lim_{x\to-2}f(x)}=\frac{-2+(-6)}{7-(-6)}$$

$$=\boxed{\frac{-8}{13}}$$

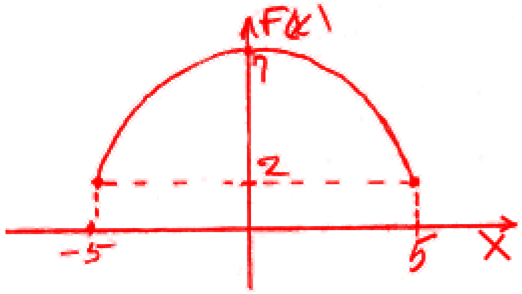
11. Sketch the function, $f(x) = \sqrt{25 - x^2} + 2$. Use the sketch to find the following limits.

$\lim_{x\to5^-} f(x) =$ **2**

$\lim_{x\to5^+} f(x) =$ **D. N. E.**

$\lim_{x\to5} f(x) =$ **D. N. E.**

$\lim_{x\to0} f(x) =$ **7**



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12. $$\lim_{x\to0}\frac{\cos^2(2x) - 1}{x} = ?$$

$$\lim_{x\to0}\frac{-(1-\cos^2(2x))}{x} = \lim_{x\to0}\frac{-(1-\cos(2x))(1+\cos(2x))}{x}\;\frac{2}{2}$$

$$=\left[\underbrace{\lim_{x\to0}\frac{1-\cos(2x)}{2x}}_{0}\right]\left[\underbrace{\lim_{x\to0}(-2(1+\cos(2x)))}_{-4}\right] = 0(-4)$$

$$=\boxed{0}$$

---

13. $$\lim_{x\to0}\frac{5x}{\sin(x)} = ?$$

$$=\lim_{x\to0}\left(5\,\frac{x}{\sin x}\right) = 5\left[\underbrace{\lim_{x\to0}\frac{x}{\sin x}}_{1}\right] = 5\cdot1 = \boxed{5}$$

---

14. $$\lim_{x\to\infty}\frac{7x^3 - 2x}{4x^3 - x} = ?$$

$$\lim_{x\to\infty}\frac{7x^3-2x}{4x^3-x}\;\frac{\frac{1}{x^3}}{\frac{1}{x^3}} = \lim_{x\to\infty}\frac{\frac{7x^3}{x^3}-\frac{2x}{x^3}}{\frac{4x^3}{x^3}-\frac{x}{x^3}}$$

$$=\lim_{x\to\infty}\frac{7-\cancel{\frac{2}{x^2}}\to0}{4-\cancel{\frac{1}{x^2}}\to0} = \boxed{\frac{7}{4}}$$

15. $$\lim_{x\to-\infty} 3\frac{5+2^x}{15-16x} = ?$$

$$= 3\lim_{x\to-\infty}\frac{5+2^x}{15-16x} = 3\,\frac{5+2^{-\infty}}{15-16(-\infty)}$$

$$= 3\,\frac{5+\frac{1}{2^{\infty}}\to 0}{15+\infty} = 3\,\frac{5+0}{\infty} = \frac{15}{\infty}$$

$$= \boxed{0}$$

---

16. $$\lim_{x\to-\infty}(4-10x^2-6x^3)$$

$$= \lim_{x\to-\infty}(-6x^3)$$

$$= -6(-\infty) = \boxed{+\infty}$$

17. At what *x* value(s) is this function discontinuous?

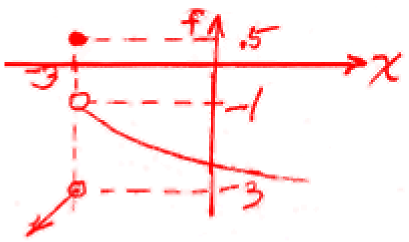
$$f(x) = \frac{x-3}{x(x+9)^2}$$

x= 0 and x = – 9

---

18. Sketch this piecewise function and then answer the questions.

$$f(x) = \begin{cases} x & when\ x < -3 \\ .5 & when\ x = -3 \\ -\sqrt{x+3}-1 & when\ x > -3 \end{cases}$$



$\lim_{x\to-3^-} f(x) =$ $\mathbf{-3}$

$\lim_{x\to-3^+} f(x) =$ $\mathbf{-1}$

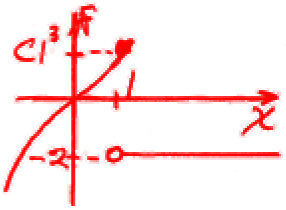
$\lim_{x\to-3} f(x) =$ **D. N. E.**

$f(-3) = .5$

---

19. Determine the value of C so as to insure that this function is everywhere continuous.

$$f(x) = \begin{cases} Cx^3 & \text{if } x \le 1 \\ -2 & \text{if } x > 1 \end{cases}$$



$$c\,1^3 = -2$$

$$c = \boxed{-2}$$

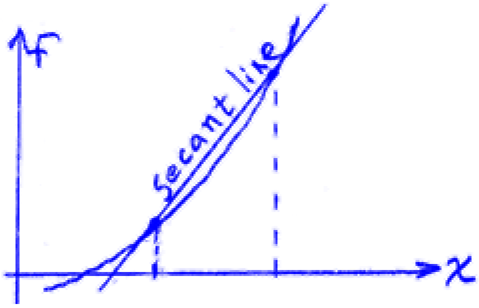
Calculus, Unit 2

Derivative fundamentals

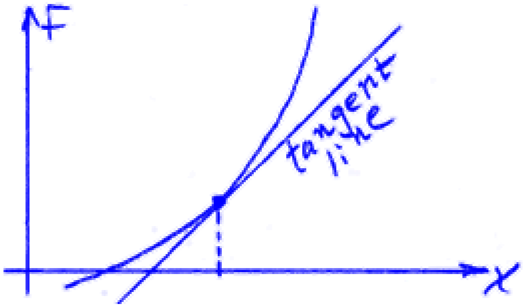
Unit 2: Lesson 01

Average and instantaneous rates of change
Definition of the derivative at x = c

The **average rate of change** between two points on a function is the **slope** of a secant line drawn between those two points.



The **instantaneous rate of change** of a function at a point on that function is the **slope** of a tangent to the curve at that point.

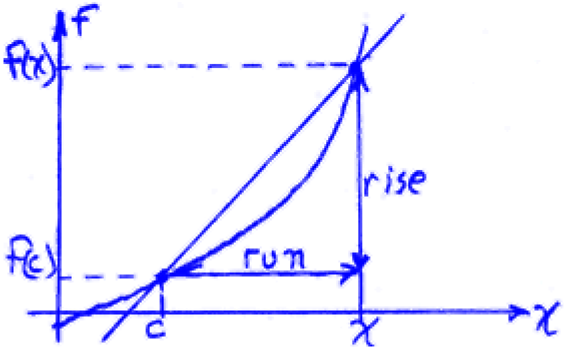


The instantaneous rate of change of a function at a point is called the **derivative** of the function at that point and is defined as a limit:

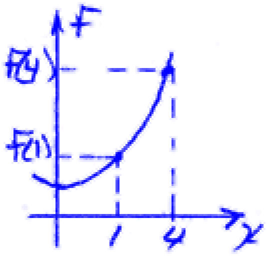
$$f'(c) = \lim_{x \to c} \frac{f(x) - f(c)}{x - c}$$

Read *f′(c)* as, "*f* prime of c" which means the "derivative of *f evaluated at c*."

$$m = \frac{rise}{run} = \frac{f(x) - f(c)}{x - c}$$



Example 1: Find the average rate of change of the function f(x) = $3x^2 + 2$ between x =1 and x =4.



$$m = \frac{f(4) - f(1)}{4 - 1} = \frac{3 \cdot 4^2 + 2 - (3 \cdot 1^2 + 2)}{3}$$

$$= \frac{48 + 2 - (3 + 2)}{3} = \frac{50 - 5}{3} = \frac{45}{3}$$

$$= \boxed{15}$$

Example 2: Find the instantaneous rate of change of the function $f(x) = 3x^2 + 2$ at $x = c = 1$.

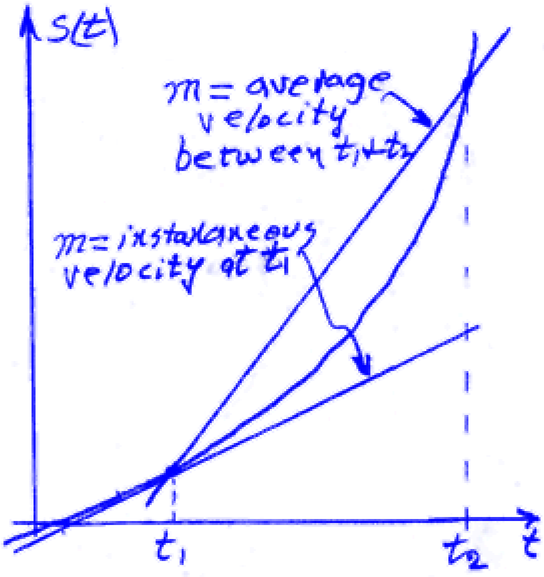
$$f'(1) = \lim_{x\to 1} \frac{f(x) - f(1)}{x-1} = \lim_{x\to 1} \frac{3x^2+2-(3\cdot 1^2+2)}{x-1}$$

$$= \lim_{x\to 1} \frac{3x^2+\cancel{2}-3\cdot 1^2-\cancel{2}}{x-1} = \lim_{x\to 1} \frac{3(x^2-1^2)}{x-1}$$

$$= \lim_{x\to 1} \frac{3\cancel{(x-1)}(x+1)}{\cancel{x-1}} = \lim_{x\to 1} 3(x+1) = 3(1+1) = \boxed{6}$$

Are these rates of change (both average and instantaneous) just mathematical abstractions, or are there "real world" applications?

If *s(t)* is the time-position of an object moving along a straight line, then the average rate of change of this function is the **average velocity** (over some time-interval) and the instantaneous rate of change is the **instantaneous velocity** at some particular time.



Example 3: Find the average velocity of the object whose time-position is given by $s(t) = t^2 - 6t - 3$ meters, between t = 2 sec and t = 6 sec.

$$V_{ar} = m = \frac{s(6)-s(2)}{6-2} = \frac{6^2-6(6)-3-(2^2-6\cdot 2-3)}{4}$$

$$= \frac{\cancel{36}-\cancel{36}-3-(4-12-3)}{4}$$

$$= \frac{-3-(-11)}{4} = \frac{-3+11}{4} = \frac{8}{4} = \boxed{2\ m/sec}$$

Example 4: Find the instantaneous velocity of the object whose time-position is given by $s(t) = t^2 - 6t - 3$ meters , at $t = 2$ sec.

$$s'(c) = \lim_{t \to 2} \frac{s(t) - s(2)}{t - 2} = \lim_{t \to 2} \frac{t^2 - 6t - 3 - (2^2 - 6 \cdot 2 - 3)}{t - 2}$$

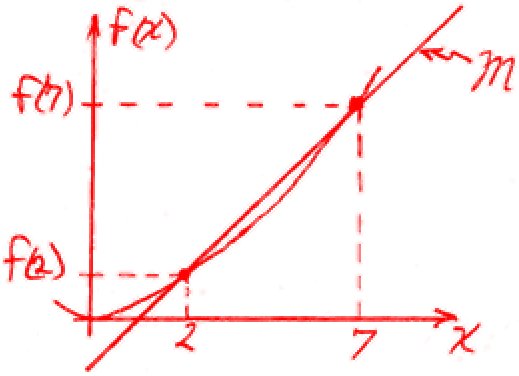
$$= \lim_{t \to 2} \frac{t^2 - 6t - 3 - (4 - 12 - 3)}{t - 2} = \lim_{t \to 2} \frac{t^2 - 6t - 3 + 11}{t - 2}$$

$$= \lim_{t \to 2} \frac{t^2 - 6t + 8}{t - 2} = \lim_{t \to 2} \frac{(t-4)\cancel{(t-2)}}{\cancel{t-2}} = \lim_{t \to 2}(t-4)$$

$$= 2 - 4 = \boxed{-2\ m/sec}$$

Assignment:

1. For the function $f(x) = 4x^2$, find the average rate of change between x = 2 and x = 7.



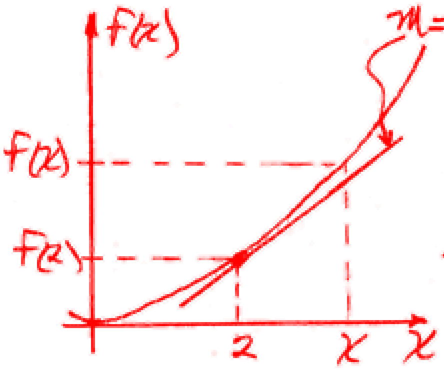
$$m = \frac{f(7)-f(2)}{7-2} = \frac{4(7)^2-4(2)^2}{7-2}$$

$$= \frac{196-16}{5} = \frac{180}{5}$$

$$= \boxed{36}$$

---

2. For the function $f(x) = 4x^2$, find the instantaneous rate of change at x = 2.



$$m = f' = \lim_{x\to c}\frac{f(x)-f(c)}{x-c} = \lim_{x\to 2}\frac{f(x)-f(2)}{x-2}$$

$$= \lim_{x\to 2}\frac{4x^2-4\cdot4}{x-2} = \lim_{x\to 2}\frac{4(x^2-4)}{x-2}$$

$$= \lim_{x\to 2}\frac{4\cancel{(x-2)}(x+2)}{\cancel{x-2}}$$

$$= \lim_{x\to 2}4(x+2) = 4(2+2) = \boxed{16}$$

---

3. For the time-position function $s(t) = t^3 + 4$ meters, find the average velocity between t = 3sec and t = 11sec.

$$V_{Av} = \frac{s(11)-s(3)}{11-3} = \frac{11^3+4-(3^3+4)}{11-3} = \frac{1331+\cancel{4}-27-\cancel{4}}{8}$$

$$= \frac{1331-27}{8} = \frac{1304}{8} = \boxed{163\ m/sec}$$

4. For the time-position function s(t) = t^3 + 4 feet, find the instantaneous velocity at t = 3 min.

$$V = \lim_{t\to3} \frac{s(t)-s(3)}{t-3} = \lim_{t\to3}\frac{t^3+4-(3^3+4)}{t-3}$$

$$= \lim_{t\to3}\frac{t^3+\cancel{4}-27-\cancel{4}}{t-3} = \lim_{t\to3}\frac{\cancel{(t-3)}(t^2+3t+9)}{\cancel{t-3}}$$

$$= 3^2+3\cdot3+9 = \boxed{27\text{ ft/min}}$$

5. What is the derivative of f(x) = –2x + 5 at x = c = 11?

$$f' = \lim_{x\to11}\frac{f(x)-f(11)}{x-11} = \lim_{x\to11}\frac{-2x+5-(-2\cdot11+5)}{x-11}$$

$$= \lim_{x\to11}\frac{-2x+\cancel{5}+22-\cancel{5}}{x-11} = \lim_{x\to11}\frac{-2\cancel{(x-11)}}{\cancel{x-11}}$$

$$= \boxed{-2}$$

6. Find f'(–2) where f(x) = –5x^2 + x –12.

$c = -2$

$$f'(-2) = \lim_{x\to-2}\frac{f(x)-f(-2)}{x-(-2)} = \lim_{x\to-2}\frac{-5x^2+x-12-(-5(-2)^2-2-12)}{x+2}$$

$$= \lim_{x\to-2}\frac{-5x^2+x-\cancel{12}+20+2+\cancel{12}}{x+2} = \lim_{x\to-2}\frac{-5x^2+x+22}{x+2}$$

$$= \lim_{x\to-2}\frac{(-5x+11)\cancel{(x+2)}}{\cancel{x+2}} = \lim_{x\to-2}(-5x+11) = -5(-2)+11$$

$$= \boxed{21}$$

7. Find the average rate of change of the function given by g(x) = $x^3 - x$ over the interval from x = –1 to x = 7.

$$m = A.R.O.C. = \frac{f(7) - f(-1)}{7-(-1)} = \frac{7^3 - 7 - ((-1)^3 - (-1))}{8}$$

$$= \frac{343 - 7 - \cancel{(-1+1)} \to 0}{8}$$

$$= \frac{336 - 0}{8} = \boxed{42}$$

---

8. What is the instantaneous velocity of an object in free-fall when its vertical position is given by s(t) = 400 –4.9t^2 meters after t = 3 seconds?

$$V = \lim_{t\to 3} \frac{s(t) - s(3)}{t-3} = \lim_{t\to 3} \frac{400 - 4.9t^2 - (400 - 4.9\cdot 3^2)}{t-3}$$

$$= \lim_{t\to 3} \frac{\cancel{400} - 4.9t^2 - \cancel{400} + 44.1}{t-3} = \lim_{t\to 3} \frac{-4.9(t^2-9)}{t-3}$$

$$= \lim_{t\to 3} \frac{-4.9\cancel{(t-3)}(t+3)}{\cancel{t-3}} = -4.9(3+3) = \boxed{-29.4 \text{ m/sec}}$$

---

9. What is f'(c) when c = 5 and f(x) = x^2 –7x + 2?

$$f'(5) = \lim_{x\to 5} \frac{f(x) - f(5)}{x-5} = \lim_{x\to 5} \frac{x^2 - 7x + 2 - (5^2 - 7\cdot 5 + 2)}{x-5}$$

$$= \lim_{x\to 5} \frac{x^2 - 7x + \cancel{2} - 25 + 35 - \cancel{2}}{x-5} = \lim_{x\to 5} \frac{x^2 - 7x + 10}{x-5}$$

$$= \lim_{x\to 5} \frac{\cancel{(x-5)}(x-2)}{\cancel{x-5}} = \lim_{x\to 5}(x-2) = 5-2 = \boxed{3}$$

10. What is the slope of the tangent line at x = –4 of the curve given by f(x) = 4x – x^2?

$$m = f'(-4) = \lim_{x\to -4}\frac{f(x)-f(-4)}{x-(-4)} = \lim_{x\to -4}\frac{4x-x^2-(-16-16)}{x+4}$$

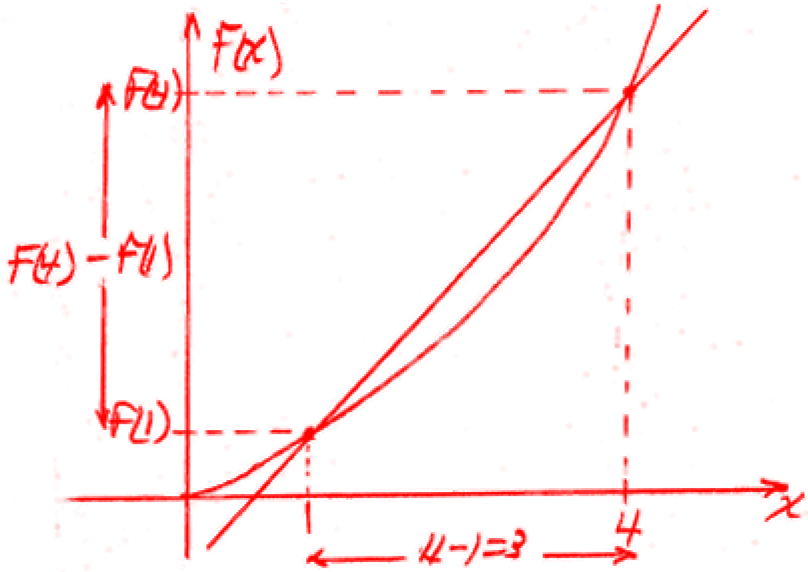
$$= \lim_{x\to -4}\frac{4x-x^2+32}{x+4} = \lim_{x\to -4}\frac{-(x^2-4x-32)}{x+4}$$

$$= \lim_{x\to -4}\frac{-\cancel{(x+4)}(x-8)}{\cancel{x+4}} = \lim_{x\to -4}(-x+8)$$

$$= -(-4)+8 = \boxed{12}$$

---

11. Draw the curve f(x) = x^2 and label all that would be necessary to find the slope of the secant line between the two points on the curve given by x = 1 and x = 4.



Unit 2: Lesson 02 Equations of tangent and normal lines

In this lesson we will find the equation of the tangent line to a curve at a particular point and also the equation of a normal (perpendicular) line at the point. To do this, use the following:

- The y-value of the point is obtained by **evaluating the function** at the given x-value.
- The **slope** of the tangent line is the **derivative** of the function at that particular x-value.
- The slope of the normal line is the **negative reciprocal** of the slope of the tangent line

Example 1: Find the equation of the tangent line to the curve f(x) at x = 3 where $f(x) = 4x^2 - x + 7$.

$$f(3) = 4\cdot 3^2 - 3 + 7$$
$$= 36 + 4$$
$$= 40$$
$$(x, y) = (3, 40)$$

$$m = f'(3) = \lim_{x\to 3}\frac{f(x) - f(3)}{x - 3}$$
$$= \lim_{x\to 3}\frac{4x^2 - x + 7 - (36 - 3 + 7)}{x - 3}$$
$$= \lim_{x\to 3}\frac{4x^2 - x - 33}{x - 3} = \lim_{x\to 3}\frac{(4x+11)\cancel{(x-3)}}{\cancel{x-3}}$$
$$= 4\cdot 3 + 11 = 23$$

$y = mx + b$ sub in (3, 40)

$y = 23x + b$

$40 = 23\cdot 3 + b$

$-29 = b$

$y = mx + b$

$$\boxed{y = 23x - 29}$$

Example 2: Find the equation of the normal line to the curve f(x) at x = 3 where $f(x) = 4x^2 - x + 7$.

$$m_{tan} = 23$$
$$m_{\perp} = -\frac{1}{23}$$

$$y = mx + b$$
$$y = -\frac{1}{23}x + b$$

sub in (3, 40)

$$40 = -\frac{3}{23} + b$$
$$\frac{23}{23}\cdot\frac{40}{1} + \frac{3}{23} = b$$
$$\frac{923}{23} = b$$

$$y = mx + b$$
$$y = \boxed{-\frac{1}{23}x + \frac{923}{23}}$$

Assignment:

1. Find the equation of the tangent line to the curve f(x) at x = –4 where f(x) = $x^2 - x + 1$.

$f(-4)=(-4)^2-(-4)+1$

$=16+4+1=21$

$(x,y)=(x,f(-4))$

$=(-4,21)$

$m=f(-4)=\lim_{x\to -4}\frac{f(x)-f(-4)}{x-(-4)}$

$=\lim_{x\to -4}\frac{x^2-x+1-(16+4+1)}{x+4}$

$=\lim_{x\to -4}\frac{x^2-x+1-21}{x+4}=\lim_{x\to -4}\frac{x^2-x-20}{x+4}$

$=\lim_{x\to -4}\frac{\cancel{(x+4)}(x-5)}{\cancel{x+4}}=-4-5=-9$

$y=mx+b$

$y=-9x+b$

sub in $(-4,21)$

$21=-9(-4)+b$

$-15=b$

$y=mx+b$

$\boxed{y=-9x-15}$

2. Find the equation of the normal line to the curve f(x) at x = –4 where f(x) = $x^2 - x + 1$.

$m_{tan}=-9$

$m_\perp=\frac{1}{9}$

$y=mx+b$

$y=\frac{1}{9}x+b$

sub in $(-4,21)$

$21=\frac{1}{9}(-4)+b$

$\frac{9}{9}\frac{21}{1}+\frac{4}{9}=b$

$\frac{193}{9}=b$

$y=mx+b$

$\boxed{y=\frac{1}{9}x+\frac{193}{9}}$

3. What is the equation of the normal line to the curve given by f(x) = 2/x at x = –1?

$m_{tan}=f'(-1)=\lim_{x\to -1}\frac{\frac{2}{x}-\left(\frac{2}{-1}\right)}{x-(-1)}=\lim_{x\to -1}\frac{\frac{2}{x}+2}{x+1}$

$=\lim_{x\to -1}\frac{\frac{2}{x}\cancel{(1+x)}}{\cancel{x+1}}=\frac{2}{-1}=-2$

$m_\perp=\frac{1}{2}$

$f(-1)=\frac{2}{-1}=-2\,;(x,y)=(-1,-2)$

$y=mx+b$

$y=\frac{1}{2}x+b$

sub in $(-1,-2)$

$-2=\frac{1}{2}(-1)+b$

$-2=-\frac{1}{2}+b$

$\frac{2}{2}\frac{-2}{1}+\frac{1}{2}=b$

$-\frac{3}{2}=b$

$y=mx+b$

$\boxed{y=\frac{1}{2}x-\frac{3}{2}}$

4. What is the equation of the tangent line to the curve given by $f(x) = \sqrt{x}$ at x = 5?

$$m_{tan} = f'(5) = \lim_{x\to5}\frac{f(x)-f(5)}{x-5} = \lim_{x\to5}\frac{\sqrt{x}-\sqrt{5}}{x-5}\,\frac{\sqrt{x}+\sqrt{5}}{\sqrt{x}+\sqrt{5}}$$

$$= \lim_{x\to5}\frac{\cancel{(x-5)}}{\cancel{x-5}\,(\sqrt{x}+\sqrt{5})} = \lim_{x\to5}\frac{1}{\sqrt{x}+\sqrt{5}} = \frac{1}{\sqrt{5}+\sqrt{5}} = \frac{1}{2\sqrt{5}}$$

$f(5) = \sqrt{5}$

$(x,y) = (5, f(5))$

$= (5, \sqrt{5})$

$y = mx+b$

$y = \frac{1}{2\sqrt{5}}x + b$

sub in $(5,\sqrt{5})$

$\sqrt{5} = \frac{5}{2\sqrt{5}} + b$

$b = \frac{\sqrt{5}}{2}$

$y = mx + b$

$$\boxed{y = \frac{1}{2\sqrt{5}}x + \frac{\sqrt{5}}{2}}$$

---

5. Find the equation of the normal line to the curve $x^2/3 + 2$ at the point (3, 5).

$$m_{tan} = f'(3) = \lim_{x\to3}\frac{f(x)-f(3)}{x-3} = \lim_{x\to3}\frac{\frac{1}{3}x^2+2-(\frac{9}{3}+2)}{x-3}$$

$$= \lim_{x\to3}\frac{\frac{1}{3}x^2+2-5}{x-3} = \lim_{x\to3}\frac{\frac{1}{3}x^2-3}{x-3} = \lim_{x\to3}\frac{\frac{1}{3}(x^2-9)}{x-3}$$

$$= \lim_{x\to3}\frac{\frac{1}{3}\cancel{(x-3)}(x+3)}{\cancel{x-3}} = \frac{1}{3}(3+3) = 2 \;/\; m_\perp = -\frac{1}{2}$$

$y = mx+b$

$y = -\frac{1}{2}x + b$

sub in $(3,5)$

$5 = -\frac{3}{2} + b$

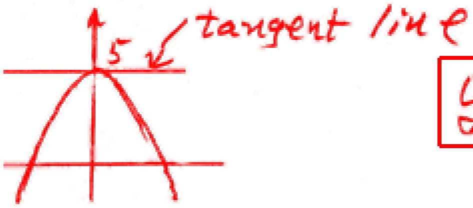
$\frac{13}{2} = b$

$y = mx + b$

$$\boxed{y = -\frac{1}{2}x + \frac{13}{2}}$$

---

6. Sketch the graph of $y = -x^2 + 5$. Without doing any mathematics and just by looking at the sketch, what would you guess the equation of the tangent at x = 0 to be?



$$\boxed{y = 5}$$

7. If *m* is the slope of the tangent line to the curve given by $f(x) = -x^2$, show that m = –8 at (4, –16).

Show that $m = f'(4) = -8$

$$f'(4) = \lim_{x\to4}\frac{f(x)-f(4)}{x-4} = \lim_{x\to4}\frac{-x^2-(-4^2)}{x-4}$$

$$= \lim_{x\to4}\frac{-x^2+16}{x-4} = \lim_{x\to4}\frac{-(x^2-16)}{x-4}$$

$$= \lim_{x\to4}\frac{-\cancel{(x-4)}(x+4)}{\cancel{x-4}} = -(4+4) = \boxed{-8}$$

---

8. If the derivative of f(x) at x = c = 2 is –5 and f(2) = 13, what is the equation of the tangent line at x = 2?

$(x,y) = (2,13)$; $m = -5$

$y = mx+b$

$y = -5x+b$

sub in $(2,13)$

$13 = -5(2)+b$

$13+10 = b$

$23 = b$

$y = mx+b$

$$\boxed{y = -5x+23}$$

---

9. Consider a parabola having its vertex at (2,1) and passing through (–4, 7). What is the equation of the tangent line at x = 8?

First, find eq. of the parabola.

$y = a(x-2)^2+1$

sub in $(-4,7)$

$7 = a(-4-2)^2+1$

$7 = a(-6)^2+1$

$a = 6/36 = \frac{1}{6}$

$y = f(x)$

$= \frac{1}{6}(x-2)^2+1$

$f(8) = 7$

$(x,y) = (8, f(8)) = (8,7)$

$$m = f'(8) = \lim_{x\to8}\frac{f(x)-f(8)}{x-8}$$

$$= \lim_{x\to8}\frac{\frac{1}{6}(x-2)^2+1-\left(\frac{1}{6}(8-2)^2+1\right)}{x-8}$$

$$= \lim_{x\to8}\frac{\frac{1}{6}(x^2-4x-32)}{x-8} = \lim_{x\to8}\frac{\frac{1}{6}\cancel{(x-8)}(x+4)}{\cancel{x-8}}$$

$$= \frac{1}{6}(8+4) = 2$$

$y-y_1 = m(x-x_1)$

$$\boxed{y-7 = 2(x-8)}$$

10. What is the equation of the normal line that passes through the vertex of the parabola described in problem 9?

Using Calculus is overkill on this problem.

The tangent line to the vertex is horizontal; therefore, the normal line is vertical. Since the coordinates of the vertex are (2, 1), the equation of the vertical normal line is x = 2.

Unit 2: Lesson 03

Formal definition of the derivative

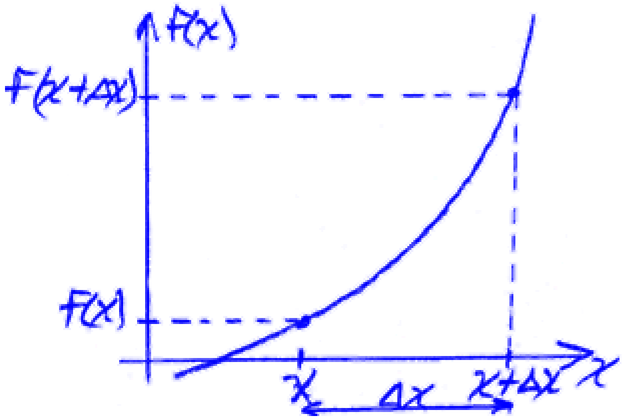
In lesson 1 of this unit, we looked at the definition of the instantaneous rate of change (the derivative) of a function at the **specific** point given by x = c.

$$f'(c) = \lim_{x \to c} \frac{f(x) - f(c)}{x - c}$$

We now present the **general** formula for the derivative of y = f(x) at the **general** position *x* and its accompanying diagram. Notice the use of *Δx* which means, “the change in *x*.”

$$\boldsymbol{f'(x) = y' = \frac{dy}{dx}}$$

$$= \lim_{\Delta x \to 0} \frac{\mathbf{f(x + \Delta x) - f(x)}}{\Delta \mathbf{x}}$$



Notice a new notation for the derivative, $\frac{dy}{dx}$. (Quite often the above formula uses *h* instead of *Δx*).

Example 1: Using the formula above find f’(x) where f(x) = $3x^2 - x$.

$$f' = \lim_{\Delta x \to 0} \frac{f(x+\Delta x) - f(x)}{\Delta x} = \lim_{\Delta x \to 0} \frac{3(x+\Delta x)^2 - (x+\Delta x) - (3x^2 - x)}{\Delta x}$$

$$= \lim_{\Delta x \to 0} \frac{3(x^2 + 2x(\Delta x) + (\Delta x)^2) - \cancel{x} - \Delta x - 3x^2 + \cancel{x}}{\Delta x}$$

$$= \lim_{\Delta x \to 0} \frac{\cancel{3x^2} + 6x(\Delta x) + 3(\Delta x)^2 - \Delta x - \cancel{3x^2}}{\Delta x}$$

$$= \lim_{\Delta x \to 0} \frac{\cancel{(\Delta x)}(6x + 3\Delta x - 1)}{\cancel{\Delta x}} = \lim_{\Delta x \to 0} (6x + 3\Delta x - 1) = \boxed{6x - 1}$$

Example 2: Use the formal definition of the derivative to find the slope of the tangent line to the curve given by $f(x) = x^2 + 6x - 2$ at $x = -4$.

$$f'(x) = \lim_{\Delta x \to 0} \frac{f(x+\Delta x) - f(x)}{\Delta x} = \lim_{\Delta x \to 0} \frac{(x+\Delta x)^2 + 6(x+\Delta x) - 2 - (x^2 + 6x - 2)}{\Delta x}$$

$$= \lim_{\Delta x \to 0} \frac{\cancel{x^2} + 2x(\Delta x) + (\Delta x)^2 + \cancel{6x} + 6(\Delta x) - \cancel{2} - \cancel{x^2} - \cancel{6x} + \cancel{2}}{\Delta x}$$

$$= \lim_{\Delta x \to 0} \frac{\cancel{\Delta x}(2x + \Delta x + 6)}{\cancel{\Delta x}}$$

$$= \lim_{\Delta x \to 0} \frac{(2x + \Delta x + 6)}{1} = 2x + 6$$

$$f'(-4) = 2(-4) + 6 = \boxed{-2} = m$$

Assignment: In the following problems, **use the new formal definition** to find the derivative of the function and **then** substitute in a particular value if asked to do so.

1. If $y = f(x) = x^2$, find $\frac{dy}{dx}$.

$$\frac{dy}{dx} = \lim_{\Delta x \to 0} \frac{f(x+\Delta x) - f(x)}{\Delta x}$$

$$= \lim_{\Delta x \to 0} \frac{(x+\Delta x)^2 - x^2}{\Delta x} = \lim_{\Delta x \to 0} \frac{\cancel{x^2} + 2x(\Delta x) + (\Delta x)^2 - \cancel{x^2}}{\Delta x}$$

$$= \lim_{\Delta x \to 0} \frac{\cancel{\Delta x}(2x + \Delta x)}{\cancel{\Delta x}} = \lim_{\Delta x \to 0} (2x + \Delta x) = \boxed{2x}$$

---

2. What is f'(x) where f(x) = (x – 5)/4 ?

$$f' = \lim_{\Delta x \to 0} \frac{f(x+\Delta x) - f(x)}{\Delta x} = \lim_{\Delta x \to 0} \frac{\frac{1}{4}(x+\Delta x) - \frac{5}{4} - \left(\frac{x}{4} - \frac{5}{4}\right)}{\Delta x}$$

$$= \lim_{\Delta x \to 0} \frac{\cancel{\frac{1}{4}x} + \frac{1}{4}(\Delta x) - \cancel{\frac{5}{4}} - \cancel{\frac{x}{4}} + \cancel{\frac{5}{4}}}{\Delta x}$$

$$= \lim_{\Delta x \to 0} \frac{\frac{1}{4}\cancel{(\Delta x)}}{\cancel{\Delta x}} = \boxed{\frac{1}{4}}$$

---

3. Evaluate y' at x = 17 where $y = 7x^2 + 2x - 1$.

$$y' = \lim_{\Delta x \to 0} \frac{f(x+\Delta x) - f(x)}{\Delta x} = \lim_{\Delta x \to 0} \frac{7(x+\Delta x)^2 + 2(x+\Delta x) - 1 - (7x^2 + 2x - 1)}{\Delta x}$$

$$= \lim_{\Delta x \to 0} \frac{\cancel{7x^2} + 14x(\Delta x) + 7(\Delta x)^2 + \cancel{2x} + 2(\Delta x) - \cancel{1} - \cancel{7x^2} - \cancel{2x} + \cancel{1}}{\Delta x}$$

$$= \lim_{\Delta x \to 0} \frac{\cancel{\Delta x}(14x + 7\Delta x + 2)}{\cancel{\Delta x}} = \lim_{\Delta x \to 0} (14x + 7\Delta x + 2)$$

$$= 14x + 2 \quad ; \quad f(17) = 14 \cdot 17 + 2 = \boxed{240}$$

4. What is the slope of the normal line to the curve given by $f(x) = \sqrt{x}$ at x = 1?

$$m_{tan} = f'(x) = \lim_{\Delta x\to 0}\frac{f(x+\Delta x)-f(x)}{\Delta x} = \lim_{\Delta x\to 0}\frac{\sqrt{x+\Delta x}-\sqrt{x}}{\Delta x}\,\frac{\sqrt{x+\Delta x}+\sqrt{x}}{\sqrt{x+\Delta x}+\sqrt{x}}$$

$$= \lim_{\Delta x\to 0}\frac{\cancel{x}+\Delta x-\cancel{x}}{\Delta x(\sqrt{x+\Delta x}+\sqrt{x})} = \lim_{\Delta x\to 0}\frac{\cancel{\Delta x}}{\cancel{\Delta x}(\sqrt{x+\Delta x}+\sqrt{x})}$$

$$= \lim_{\Delta x\to 0}\frac{1}{\sqrt{x+\Delta x}+\sqrt{x}} = \frac{1}{\sqrt{x}+\sqrt{x}} = \frac{1}{2\sqrt{x}} = m_{tan}$$

$$m_{\perp}(x) = -2\sqrt{x} \qquad m_{\perp}(1) = -2\sqrt{1} = \boxed{-2}$$

---

5. $\lim\limits_{x\to 3}\dfrac{\sqrt{x+1}-2}{x-3} = ?$

$$\lim_{x\to 3}\frac{\sqrt{x+1}-2}{x-3}\,\frac{\sqrt{x+1}+2}{\sqrt{x+1}+2}$$

$$= \lim_{x\to 3}\frac{x+1-4}{(x-3)(\sqrt{x+1}+2)}$$

$$= \lim_{x\to 3}\frac{\cancel{x-3}}{\cancel{(x-3)}(\sqrt{x+1}+2)} = \lim_{x\to 3}\frac{1}{\sqrt{x+1}+2}$$

$$= \frac{1}{\sqrt{3+1}+2} = \frac{1}{2+2} = \boxed{\frac{1}{4}}$$

---

6. What is the slope of the tangent line to the curve given by f(x) = 1/x at x = 6?

$$m = f' = \lim_{\Delta x\to 0}\frac{f(x+\Delta x)-f(x)}{\Delta x} = \lim_{\Delta x\to 0}\frac{\frac{1}{x+\Delta x}-\frac{1}{x}}{\Delta x}\,\frac{x(x+\Delta x)}{x(x+\Delta x)}$$

$$= \lim_{\Delta x\to 0}\frac{x-(x+\Delta x)}{\Delta x\;x(x+\Delta x)} = \lim_{\Delta x\to 0}\frac{\cancel{x}-\cancel{x}-\cancel{\Delta x}}{\cancel{\Delta x}\,x(x+\Delta x)}$$

$$= \lim_{\Delta x\to 0}\frac{-1}{x(x+\Delta x)} = \frac{-1}{x^2} \qquad f'(x) = \frac{-1}{x^2}$$

$$f'(6) = -\frac{1}{6^2} = \boxed{\frac{-1}{36}}$$

7. Find the equation of the tangent line to the curve f(x) = x^3 at x = – 5.

$$F'(x) = \lim_{\Delta x \to 0} \frac{f(x+\Delta x) - f(x)}{\Delta x} = \lim_{\Delta x \to 0} \frac{(x+\Delta x)^3 - x^3}{\Delta x}$$

$$= \lim_{\Delta x \to 0} \frac{\overbrace{\cancel{x^3} + 3x^2\Delta x + 3x(\Delta x)^2 + 1(\Delta x)^3}^{\text{binomial expansion}} - \cancel{x^3}}{\Delta x}$$

$$= \lim_{\Delta x \to 0} \frac{\cancel{\Delta x}(3x^2 + 3x(\Delta x) + (\Delta x)^2)}{\cancel{\Delta x}} = 3x^2 = f'(x)$$

$$f'(-5) = 3(-5)^2$$

$$m = f'(-5) = 75$$

$F(-5) = (-5)^3 = -125$

$(x, y) = (-5, f(-5))$

$= (-5, -125)$ sub in $\longrightarrow$

$y = mx + b$

$y = 75x + b$

$-125 = 75(-5) + b$

$250 = b$

$y = mx + b$

$$\boxed{y = 75x + 250}$$

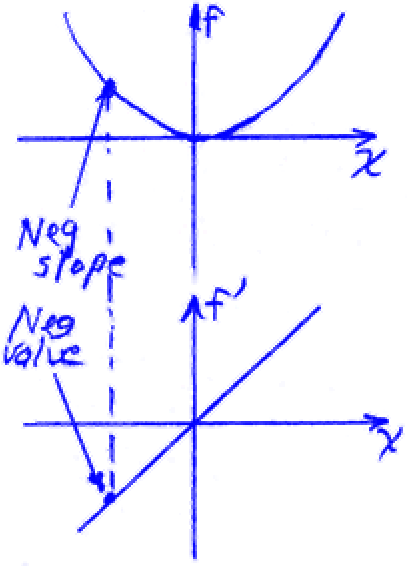
Unit 2: Lesson 04

A graphical look at derivatives

Recall that the derivative of a function at a point is really the **slope** of a tangent line at that point.

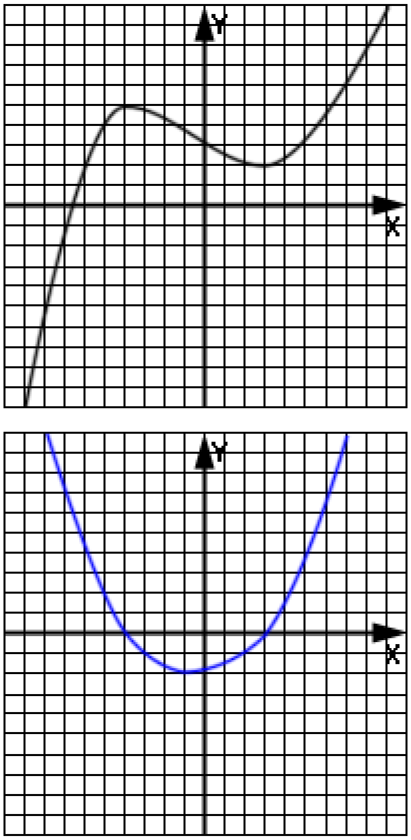
Graph both $f(x) = x^2$ and its derivative $f'(x) = 2x$ in the space provided to the right. Notice that at each corresponding x-value, *f′* is the **slope** of *f*.

It is generally true of all polynomials, that the **degree** of the derivative *f′* is **one less** than that of the original function *f*.

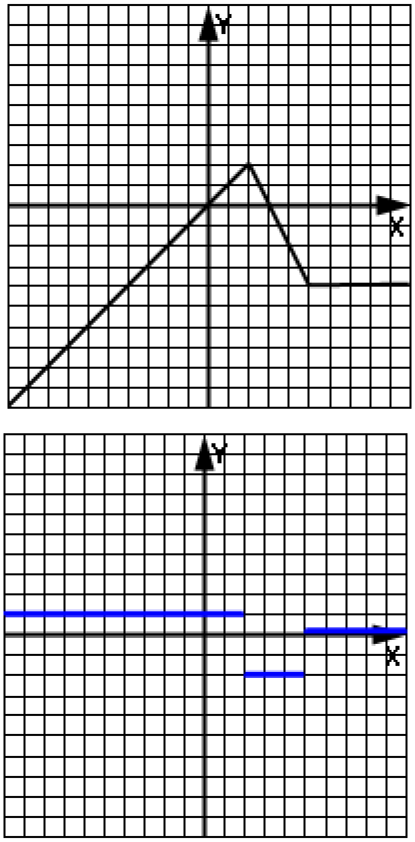


In the following two examples, consider the top graph as the original function *f(x)*. On the coordinate system just under it, sketch the graph of *f′(x)*.

Example 1:

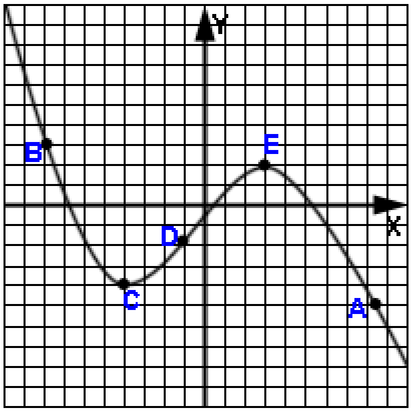


Example 2:

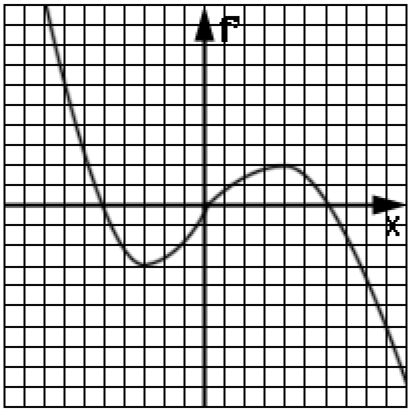


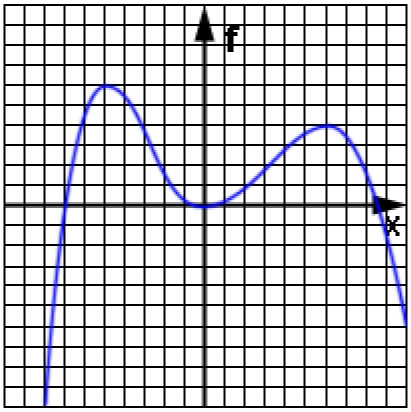
Example 3: Label directly on the graph the points as described below.

A. Point A has a negative derivative and a negative function value.
B. Point B has a positive function value and a negative derivative.
C. The slope of the tangent line at point C is 0 and the function value is negative.
D. Point D has a positive derivative.
E. Point E is a maximum point in its own little "neighborhood" and has a positive function value.

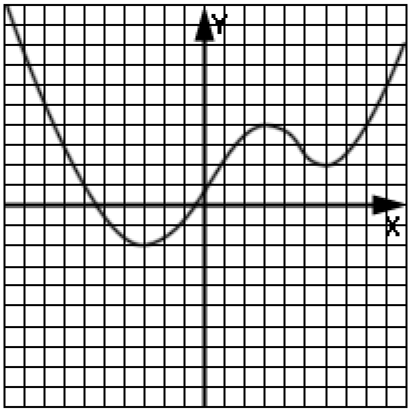


Example 4: Given that the top graph is the derivative *f′*, sketch the original function *f* on the bottom coordinate system.





Example 5: Identify the requested intervals for the function shown here.



a. Interval(s) of negative derivative
$(-\infty,-3)$, $(3, 6)$

b. Interval(s) of function decrease
$(-\infty,-3)$, $(3, 6)$

c. Interval(s) of positive derivative
$(-3, 3)$, $(6, \infty)$

d. Interval(s) of function increase
$(-3, 3)$, $(6, \infty)$

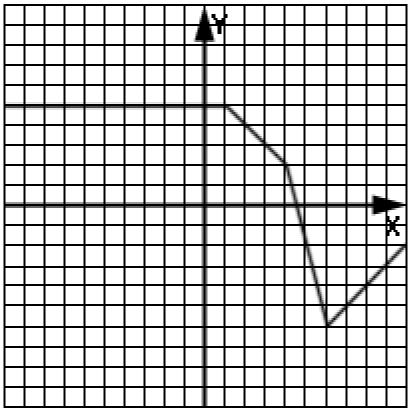
Notice from example 5 that we can infer the following:

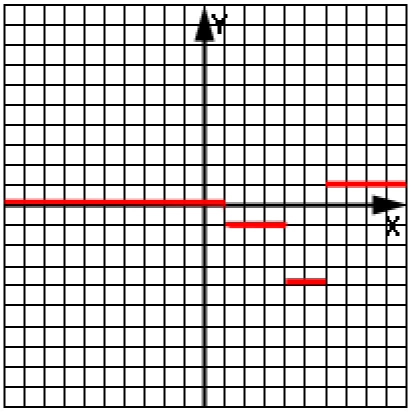
- Intervals of **negative** derivatives correspond to:
    - intervals of **negative** slope, and
    - intervals where the function is **decreasing**.

- Intervals of **positive** derivatives correspond to:
    - intervals of **positive slope**, and
    - intervals where the function is **increasing**.

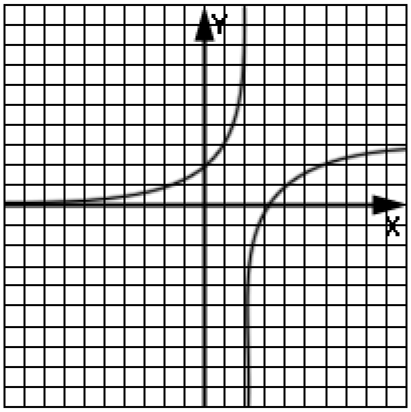
Assignment: In problems 1-4, consider the left graph as the original function *f(x)*. On the coordinate system to the right, sketch the graph of *f'(x)*.

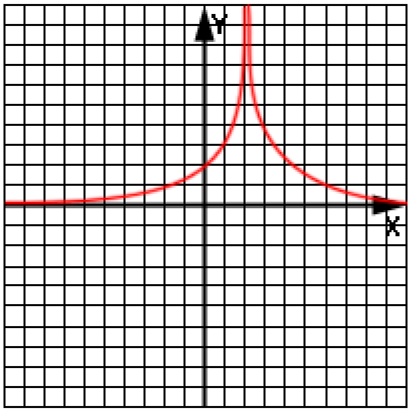
1.



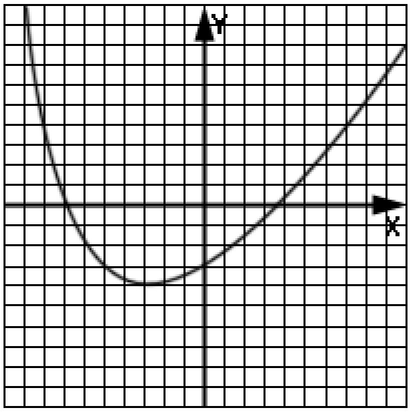


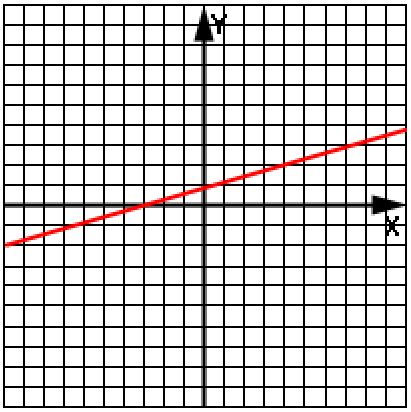
2.



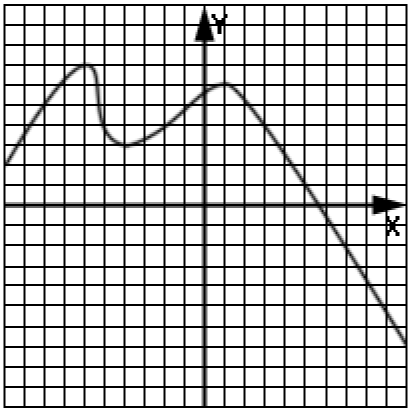


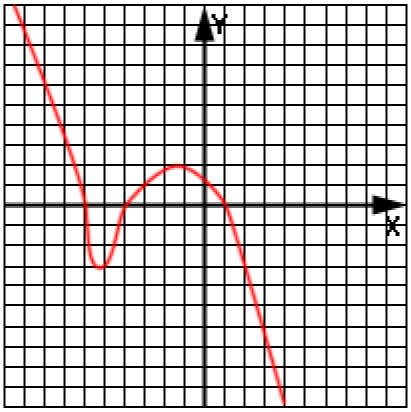
3.





4.





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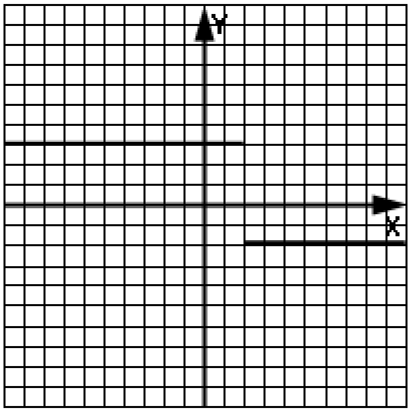
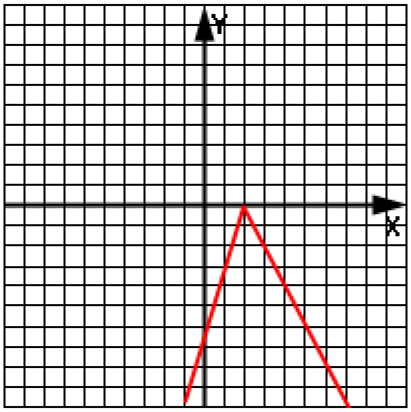
5. Separate the following six items into two associated groups of three items each: Increasing function, Decreasing function, Negative slope, Positive slope, Positive derivative, Negative derivative.

Increasing function	**Decreasing function**
Positive slope	**Negative slope**
Positive derivative	**Negative derivative**

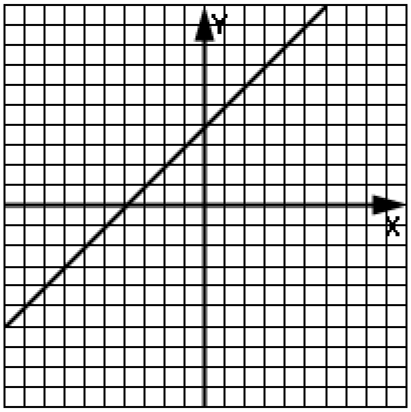
In problems 6-8, given *f′* to the left, sketch the original function *f* to the right.

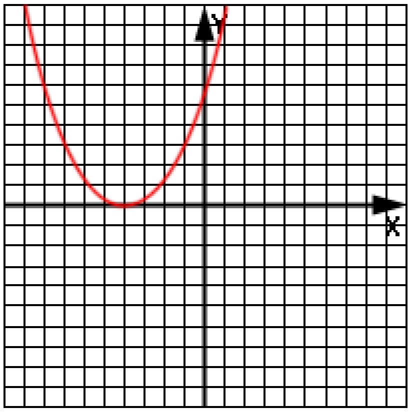
6.



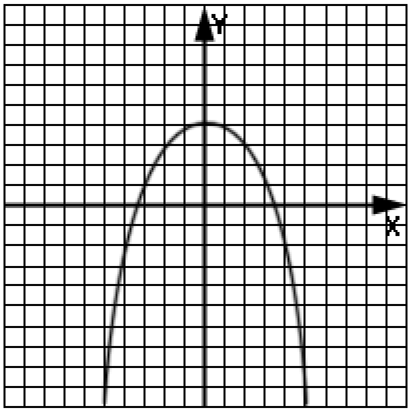


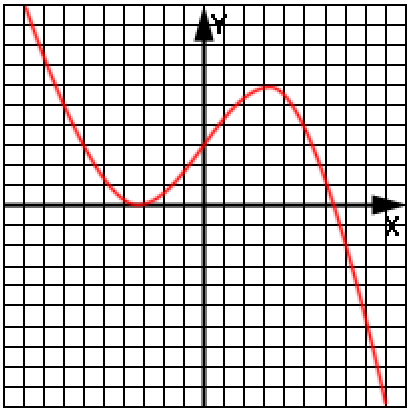
7.





8.





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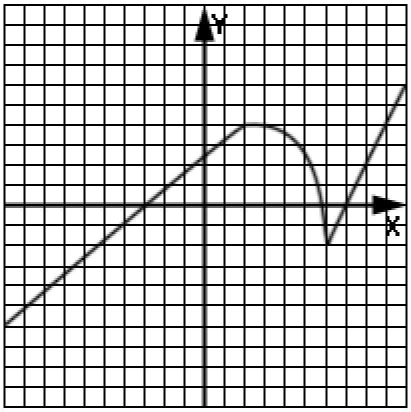
9. Identify the requested intervals for the function shown here.



a. Interval(s) of negative slope

(2, 6)

b. Interval(s) of function increase

(–∞,2), (6, ∞)

c. Interval(s) of positive derivative

(–∞,2), (6, ∞)

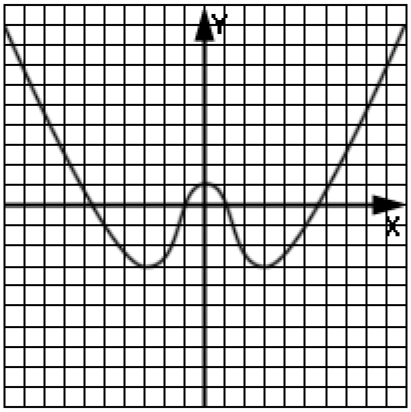
d. Interval(s) of negative derivative

(2, 6)

e. Interval(s) of positive slope

(–∞,2), (6, ∞)

10. Identify the requested intervals for the function shown here.



a. Interval(s) of positive derivative

(–3, 0), (3, ∞)

b. Interval(s) of negative slope

(–∞,–3), (0, 3)

c. Interval(s) of function increase

(–3, 0), (3, ∞)

d. Interval(s) of function decrease

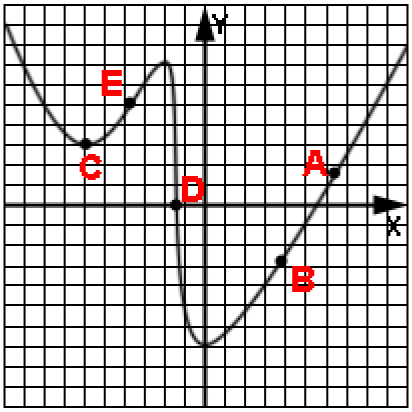
(–∞,–3), (0, 3)

e. Interval(s) of negative derivative

(–∞,–3), (0, 3)

For problems 11 and 12, label the described points directly on the graph.

11.



A. Point A has a positive derivative and a positive function value.

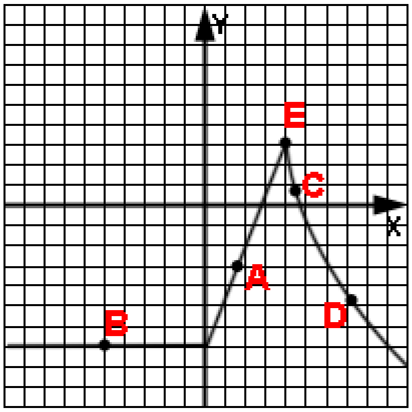
B. Point B has a negative function value and a positive derivative.

C. The slope of the tangent line at point C is 0.

D. Point D has the smallest slope of all the dots.

E. Point E has the largest function value of all the dots.

12.



A. Point A has the largest slope.

B. Point B is on an interval of the function having constant value.

C. Point C has the smallest derivative.

D. Point D has both a negative derivative and a negative function value.

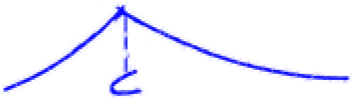
E. The slope for point E cannot be determined.

Unit 2: Lesson 05 Differentiability

A function is **not differentiable** at x = c if any of the following are true:

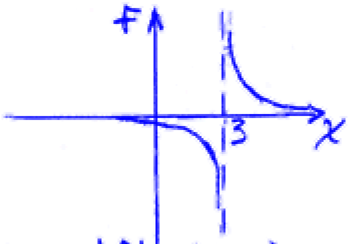
- the function is **discontinuous** at x = c,

- the function has a **cusp** (a sharp turn) at x = c, or



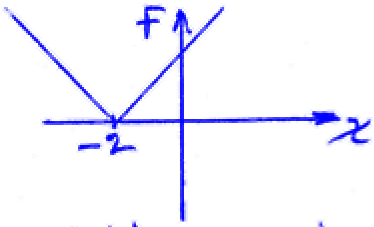
- the function has a vertical tangent line at x = c.

Example 1: Sketch the graph of f(x) = 1/(x – 3) and by visual inspection determine any point(s) at which it is not differentiable.



Not diff. (Disc)
at $x=3$

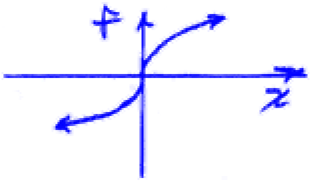
Example 2: Sketch the graph of f(x) = |x + 2| and by visual inspection determine any point(s) at which it is not differentiable.



Not diff. (cusp)
at $x=-2$

Example 3: Sketch the graph of $f(x) = 4\sqrt[3]{x}$ and by visual inspection determine any point(s) at which it is not differentiable.

$F = 4x^{1/3}$



Not diff at
$x=0$ (vertical tangent line)

Example 4: Determine if the function below is differentiable at x = 2.

$$f(x) = \begin{cases} x^2 & \text{if } x \le 2 \\ 2x & \text{if } x > 2 \end{cases}$$

$$\left.\begin{array}{l} x^2 \rightarrow 2^2 = 4 \\ 2x \rightarrow 2\cdot 2 = 4 \end{array}\right\} \checkmark \text{continuos}$$

Left side:

$$f'(2) = \lim_{x \to 2^-} \frac{f(x) - f(2)}{x-2}$$

$$f'(2) = \lim_{x \to 2^-} \frac{x^2 - (2^2)}{x-2}$$

$$f'(2) = \lim_{x \to 2^-} \frac{\cancel{(x-2)}(x+2)}{\cancel{x-2}}$$

$$= 2+2 = 4$$

Right side:

$$f'(2) = \lim_{x \to 2^+} \frac{f(x) - f(2)}{x-2}$$

$$= \lim_{x \to 2^+} \frac{2x - (2\cdot 2)}{x-2}$$

$$= \lim_{x \to 2^+} \frac{2\cancel{(x-2)}}{\cancel{x-2}}$$

$$= 2$$

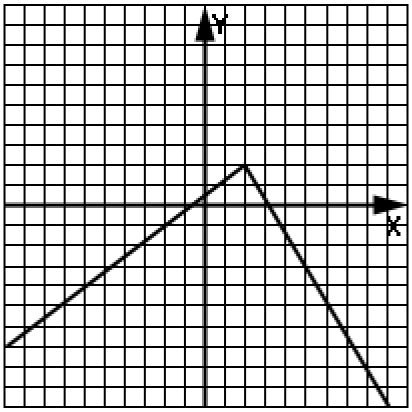
(cusp) Different answers

Limit does not exist

$\boxed{\text{Not diff at } x=2}$

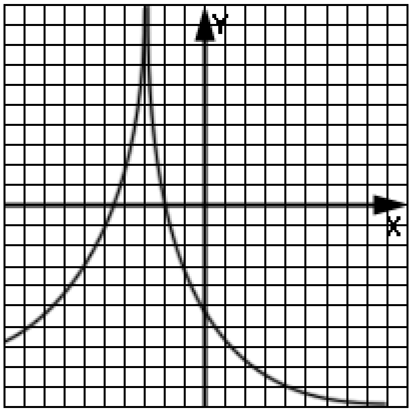
Assignment: In problems 1-6, determine any x-value(s) at which the function is not differentiable and state the reason for non-differentiability.

1.



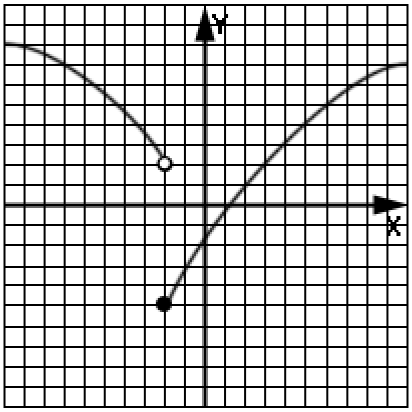
x = 2, cusp

2.



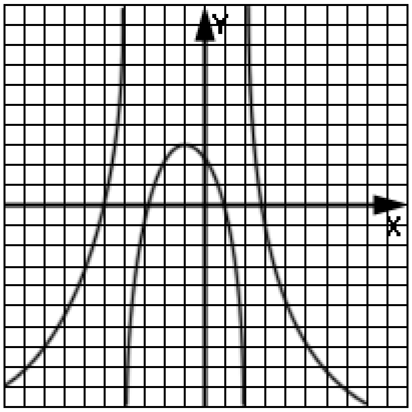
x = –3, discontinuity

3.



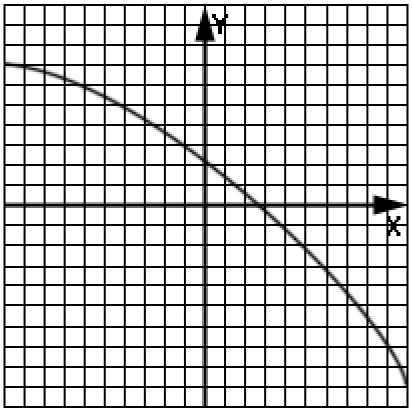
x = –2, discontinuity

4.



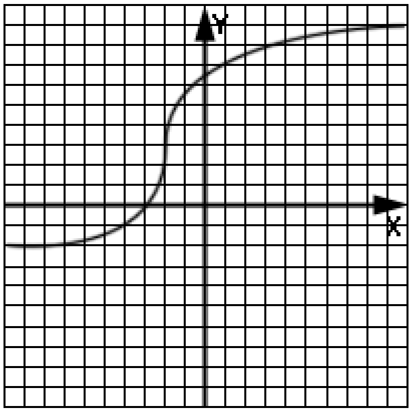
x = –4, 2, discontinuities

5.



Diff. everywhere

6.



x = –2, vertical tangent

7. Determine if the function below is differentiable at x = 4.

$$f(x) = \begin{cases} x^2 & \text{if } x \le 4 \\ 4x & \text{if } x > 4 \end{cases}$$

$$\left.\begin{array}{l} x^2 \to 4^2 = 16 \\ 4x \to 4\cdot 4 = 16 \end{array}\right\} \checkmark \text{continuous at } x = 4$$

Left side:

$$f'(4) = \lim_{x\to 4^-} \frac{f(x) - f(4)}{x-4}$$

$$= \lim_{x\to 4^-} \frac{x^2 - 4^2}{x-4}$$

$$= \lim_{x\to 4^-} \frac{\cancel{(x-4)}(x+4)}{\cancel{x-4}}$$

$$= 8$$

Right side:

$$f'(4) = \lim_{x\to 4^+} \frac{f(x) - f(4)}{x-4}$$

$$f(4) = \lim_{x\to 4^+} \frac{4x - 4\cdot 4}{x-4}$$

$$f'(4) = \lim_{x\to 4^+} \frac{4\cancel{(x-4)}}{\cancel{x-4}}$$

$$= 4$$

(cusp)

Different answers

Derivative doesn't exist

$\boxed{\text{Not diff. at } x = 4}$

---

8. Determine if the function below is differentiable at x = 1.

$$f(x) = \begin{cases} x^2 - 1 & \text{if } x \le 1 \\ x - 1 & \text{if } x > 1 \end{cases}$$

$$\left.\begin{array}{l} x^2 - 1 \to 1^2 - 1 = 0 \\ x - 1 \to 1 - 1 = 0 \end{array}\right\} \text{continuous at } x = 1$$

Left side:

$$f'(1) = \lim_{x\to 1^-} \frac{f(x) - f(1)}{x-1}$$

$$= \lim_{x\to 1^-} \frac{x^2 - 1 - \cancelto{0}{(1^2 - 1)}}{x-1}$$

$$= \lim_{x\to 1^-} \frac{\cancel{(x-1)}(x+1)}{\cancel{x-1}}$$

$$= 1 + 1 = 2$$

Right side:

$$f'(1) = \lim_{x\to 1^+} \frac{f(x) - f(1)}{x-1}$$

$$= \lim_{x\to 1^+} \frac{x - 1 - \cancelto{0}{(1-1)}}{x-1}$$

$$= \lim_{x\to 1^+} \frac{\cancel{x-1}}{\cancel{x-1}}$$

$$= 1$$

Different answers (cusp)

Derivative doesn't exist

$\boxed{\text{Not diff at } x = 1}$

9. Determine if the function below is differentiable at x = 2.

$$f(x)=\begin{cases} x^2+2 & \text{if } x\le 2 \\ 4x-2 & \text{if } x>2 \end{cases}$$

$\left.\begin{array}{l} x^2+2 \to 2^2+2=6 \\ 4x-2 \to 4\cdot2-2=6 \end{array}\right\}$ continuous ✓

Left side:

$f'(2)=\lim_{x\to2^-}\frac{f(x)-f(2)}{x-2}$

$=\lim_{x\to2^-}\frac{x^2+2-\cancelto{6}{(2^2+2)}}{x-2}$

$=\lim_{x\to2^-}\frac{x^2-4}{x-2}$

$=\lim_{x\to2^-}\frac{\cancel{(x-2)}(x+2)}{\cancel{x-2}}=4$

Right side:

$f'(2)=\lim_{x\to2^+}\frac{f(x)-f(2)}{x-2}$

$=\lim_{x\to2^+}\frac{4x-2-\cancelto{6}{(4\cdot2-2)}}{x-2}$

$=\lim_{x\to2^+}\frac{4x-8}{x-2}$

$=\lim_{x\to2^+}\frac{4\cancel{(x-2)}}{\cancel{x-2}}=4$

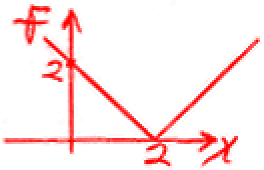
They agree!

$\boxed{\text{It is diff. at } x=2}$

Since the derivative at x = 2 from the left is the same as from the right, the function is "smooth" there (no cusp).

---

10. Determine by analysis if f(x) = |x – 2| is differentiable at x = 2. (Hint: convert to a piecewise function and then compare the left and right derivatives.)



$f(x)=\begin{cases} -x+2 & x\le 2 \\ x-2 & x>2 \end{cases}$ piecewise equivalent

Left:

$f'(2)=\lim_{x\to2^-}\frac{f(x)-f(2)}{x-2}$

$=\lim_{x\to2^-}\frac{-x+2-\cancelto{0}{(-2+2)}}{x-2}$

$=\lim_{x\to2^-}\frac{-1\cancel{(x-2)}}{\cancel{x-2}}=-1$

right:

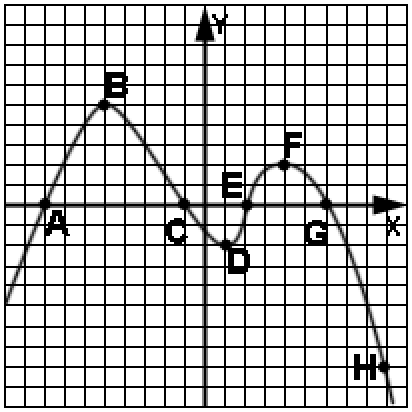
$f'(2)=\lim_{x\to2^+}\frac{f(x)-f(2)}{x-2}$

$=\lim_{x\to2^+}\frac{x-2-\cancelto{0}{(2-2)}}{x-2}$

$=\lim_{x\to2^+}\frac{\cancel{x-2}}{\cancel{x-2}}=1$

(cusp) They don't agree

$\boxed{\text{Not diff. at } x=2}$



11. Use the labeled points on the graph of this function to answer the questions below.

a. Which point(s) are roots?

A, C, E, G

b. Which point has the largest derivative?

E

c. Which point has the smallest derivative?

H

d. At which point(s) is the slope of the tangent line equal to zero?

B, D, F

e. At which point(s) is there a vertical tangent line?

None

f. Which point is the largest function value?

B

g. Which point is the smallest function value?

H

h. What is the degree of the graphed polynomial?

4 (has 4 roots)

i. What would be the degree of the derivative of the polynomial whose graph is shown here?

3 (always one degree less)

Unit 2: Review

1. Find the instantaneous rate of change of f(x) = $7x^2 - 3x$ at x = 2.

$$f'(2) = \lim_{x\to 2}\frac{f(x)-f(2)}{x-2} = \lim_{x\to 2}\frac{7x^2-3x-\cancel{(7\cdot 2^2-3\cdot 2)} \rightarrow 22}{x-2}$$

$$= \lim_{x\to 2}\frac{7x^2-3x-22}{x-2} = \lim_{x\to 2}\frac{\cancel{(x-2)}(7x+11)}{\cancel{x-2}}$$

$$= \lim_{x\to 2}(7x+11) = 7\cdot 2+11 = \boxed{25}$$

2. Find the average velocity of the time-position function s(t) =$7t^2 - 3t$ meters between t = 5 sec and t = 8 sec.

$$V_{AV} = \frac{s(8)-s(5)}{8-5}$$

$$= \frac{7\cdot 8^2-3\cdot 8-(7\cdot 5^2-3\cdot 5)}{3}$$

$$= \frac{448-24-(175-15)}{3} = \frac{448-24-160}{3}$$

$$= \frac{264}{3} = \boxed{88\ m/sec}$$

3. What is the instantaneous velocity at t = 5 sec of the time-position function s(t) = $t^3 - 5$ meters?

$$V(5) = \lim_{t\to 5}\frac{s(t)-s(5)}{t-5} = \lim_{t\to 5}\frac{t^3-5-\cancel{(5^3-5)} \rightarrow 120}{t-5}$$

$$= \lim_{t\to 5}\frac{t^3-125}{t-5} = \lim_{t\to 5}\frac{\cancel{(t-5)}(t^2+5t+25)}{\cancel{t-5}}$$

$$= \lim_{t\to 5}(t^2+5t+25) = 5^2+5\cdot 5+25 = \boxed{75\ m/sec}$$

4. Find the equation of the tangent line to the curve given by f(x) = x^2 + 7x – 17 at x = –5.

$$m=\lim_{x\to -5}\frac{f(x)-f(-5)}{x-(-5)}=\lim_{x\to -5}\frac{x^2+7x-17-\cancelto{-27}{((-5)^2+7(-5)-17)}}{x+5}$$

$$=\lim_{x\to -5}\frac{x^2+7x+10}{x+5}=\lim_{x\to -5}\frac{\cancel{(x+5)}(x+2)}{\cancel{x+5}}=-5+2=-3$$

$$f(-5)=(-5)^2+7(-5)-17$$
$$=-27$$
$$(x,y)=(-5,-27)$$

$$y-y_1=m(x-x_1)$$
$$\boxed{y+27=-3(x+5)}$$

---

5. What is the equation of the normal line at x = –5 of the curve given by f(x) = x^2 + 7x – 17?

$$\left.\begin{array}{l} m_{tan}=-3 \\ (x,y)=(-5,-27) \end{array}\right\}\text{From \#4}$$

$$m_{\perp}=\frac{1}{3}$$

$$y-y_1=m(x-x_1)$$
$$\boxed{y+27=\frac{1}{3}(x+5)}$$

---

6. Find the equation of the normal line at x = 1 of the function $f(x)=\sqrt{x}+3$.

$$m_{tan}=\lim_{x\to 1}\frac{f(x)-f(1)}{x-1}=\lim_{x\to 1}\frac{\sqrt{x}+3-(\sqrt{1}+3)}{x-1}$$

$$=\lim_{x\to 1}\frac{\sqrt{x}-1}{x-1}\cdot\frac{\sqrt{x}+1}{\sqrt{x}+1}=\lim_{x\to 1}\frac{\cancel{x-1}}{\cancel{(x-1)}(\sqrt{x}+1)}=\lim_{x\to 1}\frac{1}{\sqrt{x}+1}=\frac{1}{2}$$

$$m_{\perp}=-2$$
$$f(1)=\sqrt{1}+3$$
$$=4$$
$$(x,y)=(1,4)$$

$$y-y_1=m(x-x_1)$$
$$\boxed{y-4=-2(x-1)}$$

7. If y = 3x^2 + 5x – 1 what is $\frac{dy}{dx}$?

$$\frac{dy}{dx} = \lim_{\Delta x \to 0} \frac{f(x+\Delta x) - f(x)}{\Delta x} = \lim_{\Delta x \to 0} \frac{3(x+\Delta x)^2 + 5(x+\Delta x) - 1 - (3x^2+5x-1)}{\Delta x}$$

$$= \lim_{\Delta x \to 0} \frac{\cancel{3x^2} + 6x(\Delta x) + 3(\Delta x)^2 + \cancel{5x} + 5(\Delta x) \cancel{-1} - \cancel{3x^2} - \cancel{5x} \cancel{+1}}{\Delta x}$$

$$= \lim_{\Delta x \to 0} \frac{6x(\Delta x) + 3(\Delta x)^2 + 5(\Delta x)}{\Delta x} = \lim_{\Delta x \to 0} \frac{\cancel{\Delta x}(6x + 3(\Delta x) + 5)}{\cancel{\Delta x}} = \boxed{6x+5}$$

---

8. Find the derivative of $f(x) = \sqrt{x+2} - 3$ at x = 7.

$$f(x) = \lim_{\Delta x \to 0} \frac{f(x+\Delta x) - f(x)}{\Delta x} = \lim_{\Delta x \to 0} \frac{\sqrt{x+\Delta x+2} - 3 - (\sqrt{x+2} - 3)}{\Delta x}$$

$$= \lim_{\Delta x \to 0} \frac{\sqrt{x+\Delta x+2} - \sqrt{x+2}}{\Delta x} \cdot \frac{\sqrt{x+\Delta x+2} + \sqrt{x+2}}{\sqrt{x+\Delta x+2} + \sqrt{x+2}}$$

$$= \lim_{\Delta x \to 0} \frac{\cancel{x} + \cancel{\Delta x} + \cancel{2} - \cancel{x} - \cancel{2}}{\cancel{\Delta x}(\sqrt{x+\Delta x+2} + \sqrt{x+2})} = \lim_{\Delta x \to 0} \frac{1}{\sqrt{x+\Delta x+2} + \sqrt{x+2}}$$

$$= \frac{1}{\sqrt{x+2} + \sqrt{x+2}} = \frac{1}{2\sqrt{x+2}} \qquad f'(7) = \frac{1}{2\sqrt{7+2}} = \boxed{\frac{1}{6}}$$

---

9. Use the formal definition of the derivative to find the slope of the normal line to the curve f(x) = 1/(x + 1) at x = –4.

$$m_{tan} = \lim_{\Delta x \to 0} \frac{f(x+\Delta x) - f(x)}{\Delta x} = \lim_{\Delta x \to 0} \frac{\frac{1}{x+\Delta x+1} - \frac{1}{x+1}}{\Delta x} \cdot \frac{(x+\Delta x+1)(x+1)}{(x+\Delta x+1)(x+1)}$$

$$= \lim_{\Delta x \to 0} \frac{x+1 - (x+\Delta x+1)}{\Delta x(x+\Delta x+1)(x+1)} = \lim_{\Delta x \to 0} \frac{\cancel{x} \cancel{+1} - \cancel{x} - \Delta x \cancel{-1}}{\Delta x(x+\Delta x+1)(x+1)}$$

$$= \lim_{\Delta x \to 0} \frac{-\cancel{\Delta x}}{\cancel{\Delta x}(x+\Delta x+1)(x+1)} = \lim_{\Delta x \to 0} \frac{-1}{(x+\Delta x+1)(x+1)} = \frac{-1}{(x+1)^2}$$

$$m_{\perp}(x) = (x+1)^2; \quad m_{\perp}(-4) = (-4+1)^2 = \boxed{9}$$

10. Find the function for the velocity where the time-position function is given by s(t) = t – t^2 feet. (*t* is given in minutes).

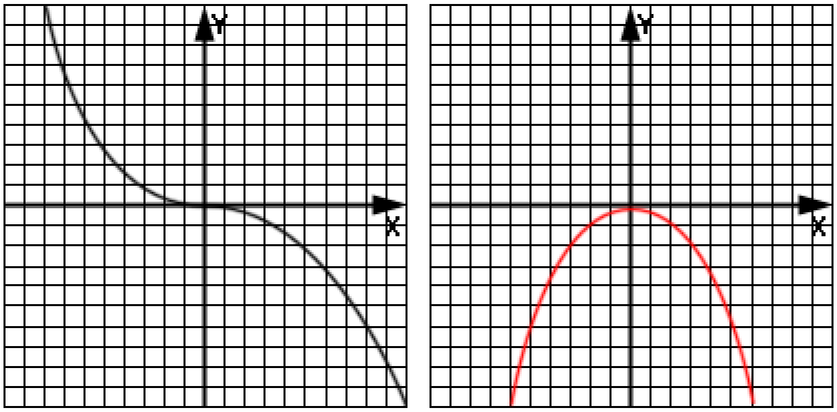
$$V(t) = \lim_{\Delta t \to 0} \frac{f(t+\Delta t) - f(t)}{\Delta t} = \lim_{\Delta t \to 0} \frac{t+\Delta t - (t+\Delta t)^2 - (t - t^2)}{\Delta t}$$

$$= \lim_{\Delta t \to 0} \frac{\cancel{t} + \Delta t - \cancel{t^2} - 2t(\Delta t) - (\Delta t)^2 - \cancel{t} + \cancel{t^2}}{\Delta t}$$

$$= \lim_{\Delta t \to 0} \frac{\cancel{\Delta t}(1 - 2t - \Delta t)}{\cancel{\Delta t}} = \lim_{\Delta t \to 0} (1 - 2t + \Delta t) = \boxed{1 - 2t \text{ ft/min}}$$

---

11. The left picture is the function *f*. Sketch its derivative *f′* on the right coordinate system.



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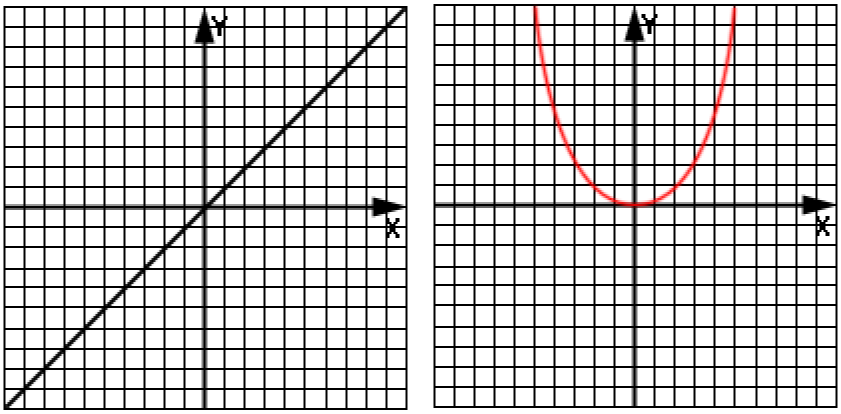
12. The left picture is the function *f′*. Sketch the original function *f* on the right coordinate system.



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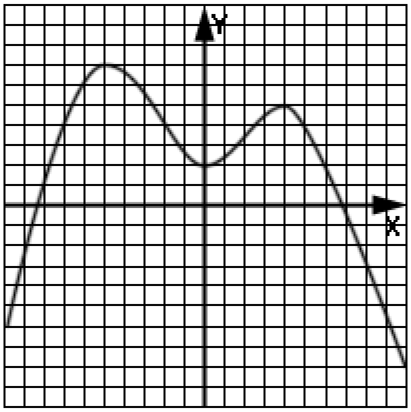
13. Separate the following six items into two associated groups of three items each: Increasing function, Decreasing function, Negative slope, Positive slope, Positive derivative, Negative derivative.

Increasing function	**Decreasing function**
Positive slope	**Negative slope**
Positive derivative	**Negative derivative**

14. Using the function whose graph is shown to the right, specify the following intervals:



a. Interval(s) of negative derivative

(–5, 0), (4, ∞)

b. Interval(s) of positive slope

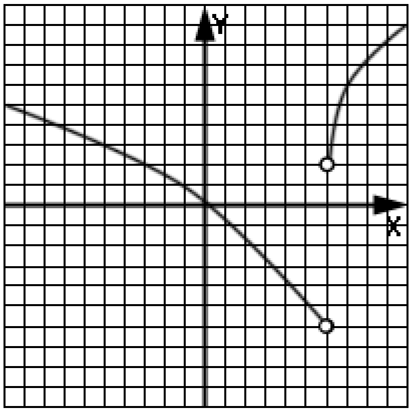
(–∞,–5), (0, 4)

c. Interval(s) of decrease

(–5, 0), (4, ∞)

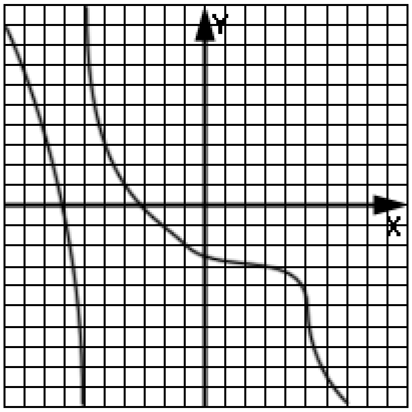
In problems 15-18, determine the point(s) at which the function is not differentiable. State the reason why.

15.



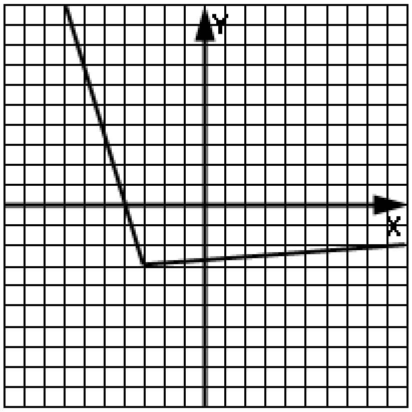
x = 6, discontinuity

16.



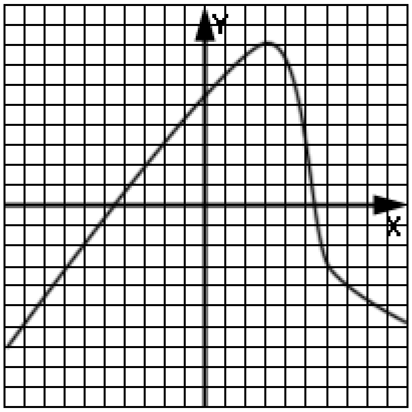
x = –6, disc; x = 5, vert tan

17.



x = –3, cusp

18.



Differentiable everywhere

19. Determine by an analysis of a continuity test and "left" & "right" derivatives if this function is differentiable at x = –2.

$$f(x) = \begin{cases} 5x^2 & \text{if } x \le -2 \\ -20x - 20 & \text{if } x > -2 \end{cases}$$

$$\left.\begin{array}{l} 5(-2)^2 \rightarrow 20 \\ -20(-2)-20 \rightarrow 20 \end{array}\right\} \checkmark \text{Continuous}$$

Left:

$$\lim_{x \to -2^-} \frac{f(x) - f(-2)}{x-(-2)}$$

$$= \lim_{x \to -2^-} \frac{5x^2 - (5(-2)^2)}{x+2}$$

$$= \lim_{x \to -2^-} \frac{5(x^2-4)}{x+2} = \lim_{x \to -2^-} \frac{5(x-2)\cancel{(x+2)}}{\cancel{x+2}}$$

$$= 5(-2-2) = -20$$

Right:

$$\lim_{x \to -2^+} \frac{f(x) - f(-2)}{x-(-2)}$$

$$= \lim_{x \to -2^+} \frac{-20x - 20 - (-20(-2)-20)}{x+2}$$

$$= \lim_{x \to -2^+} \frac{-20x - 40}{x+2}$$

$$= \lim_{x \to -2^+} \frac{-20\cancel{(x+2)}}{\cancel{x+2}} = -20$$

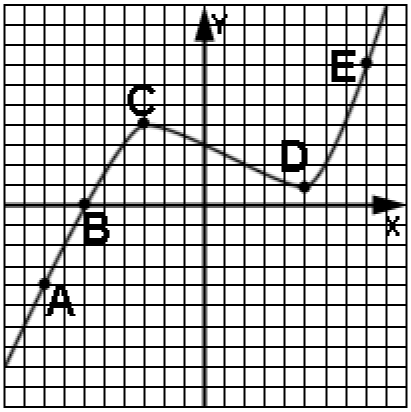
same answers

$\boxed{\text{Diff at } x = -2}$

Showing that the derivative from the left at x = -2 is equal to the derivative from the right demonstrates that the curve is "smooth" there (no cusp).

---

20. Use the letters associated with the points on this function to answer the questions.



a. The point(s) at which the tangent line is horizontal.

C, D

b. The point(s) at which the function has a root.

B

c. The point(s) at which the function value is negative and the derivative is positive.

A

d. The point(s) at which the function value is positive and the slope of the tangent line is positive.

E

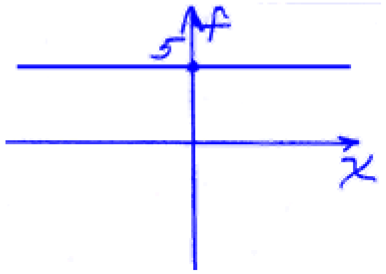
Calculus, Unit 3

Derivatives formulas
Derivative of trig and piecewise functions

Unit 3: Lesson 01

Constant and power rules

Consider the constant function f(x) = 5. Clearly the slope is 0 at every point on this “curve”, so f’(x) =0.



Derivative of a constant:

$f(x) = c$;where c is a constant

$\mathbf{f'(x) = 0}$

Power rule:

$f(x) = x^n$

$\mathbf{f'(x) = nx^{n-1}}$;where *n* can be a positive integer, a negative integer, or fractional

See **Enrichment topic C** for verification of the power rule.

Miscellaneous rules:

Because of the rules for limits and since derivatives are fundamentally based on limits, the following rules are easily produced:

If $f(x) = cg(x)$, then $\mathbf{f' = cg'}$;where *c* is a constant
if $f(x) = g(x) \pm h(x)$ then $\mathbf{f' = g' \pm h'}$

In each of examples 1-4, find the derivative of the given function.

Example 1: $f(x) = -7x^{1/2} + 22$

$$F' = -7\left(\frac{1}{2}\right)x^{-\frac{1}{2}} + 0$$

$$f'(x) = \boxed{-\frac{7}{2}x^{-1/2}}$$

Example 2: $f(x) = 3x + 2x^{-5} + 11$

$$f'(x) = 3 - 10x^{-6} + 0$$

$$= \boxed{3 - \frac{10}{x^6}}$$

Example 3: $y = \sqrt{t^3} - t$

$$y = (t^3)^{1/2} - t$$

$$y = t^{3/2} - t$$

$$y' = \boxed{\frac{3}{2}t^{\frac{1}{2}} - 1}$$

Example 4: $g(\alpha) = \dfrac{4}{\alpha^2}$

$$g(\alpha) = 4\alpha^{-2}$$

$$g'(\alpha) = -8\alpha^{-3}$$

$$= \boxed{\frac{-8}{\alpha^3}}$$

Example 5: Determine all of the *x* values of the function $f(x) = (1/3)x^3 + x^2 - 35x$ at which tangent lines are horizontal.

$$f'(x) = \frac{1}{3}(3)x^2 + 2x - 35 = 0 \leftarrow \text{slope of horiz tangent line.}$$

$$x^2 + 2x - 35 = 0$$

$$(x+7)(x-5) = 0$$

$$x + 7 = 0 \qquad x - 5 = 0$$

$$\boxed{x = -7 \qquad x = 5}$$

Assignment: In each of problems 1-6, find the derivative of the given function.

1. $f(x) = 18$

$f' = \boxed{0}$

2. $f(x) = x^4 - x^2 + 1$

$F'(x) = \boxed{4x^3 - 2x}$

3. $g(x) = \sqrt[3]{x} - 15x$

$g(x) = x^{1/3} - 15x$

$g'(x) = \frac{1}{3}x^{-2/3} - 15$

$= \boxed{\frac{1}{3x^{2/3}} - 15}$

4. $P(x) = \frac{2}{\sqrt[3]{x}}$

$P(x) = \frac{2}{x^{1/3}} = 2x^{-1/3}$

$P'(x) = 2(-\frac{1}{3})x^{-4/3}$

$= \boxed{\frac{-2}{3x^{4/3}}}$

5. $h(x) = (4x^4 - x^3 + x)/x$

$h(x) = \frac{\cancel{x}(4x^3 - x^2 + 1)}{\cancel{x}}$

$h'(x) = \boxed{12x^2 - 2x}$

6. $y = 5t^0 - 7t^3 + t$

$y = 5 \cdot 1 - 7t^3 + t$

$y = 5 - 7t^3 + t$

$y' = \boxed{-21t^2 + 1}$

7. Find the equation of the tangent line to the curve $y = x^3 - 8x^2 + x - 1$ at $x = 3$.

$m = y' = 3x^2 - 16x + 1$

$y'(3) = 3\cdot 3^2 - 16\cdot 3 + 1$

$= 27 - 48 + 1$

$= -20$

$y(3) = 3^3 - 8\cdot 3^2 + 3 - 1$

$= 27 - 72 + 2$

$= -43$

$(x, y) = (3, -43)$

$y - y_1 = m(x - x_1)$

$\boxed{y + 43 = -20(x - 3)}$

---

8. What is the equation of the normal line to $f(x) = 11/x$ at $x = -4$?

$f(x) = 11x^{-1}$

$m_{tan} = f' = -11x^{-2} = \frac{-11}{x^2}$

$f'(-4) = \frac{-11}{(-4)^2} = \frac{-11}{16}$

$m_\perp = \frac{16}{11}$

$f(-4) = \frac{11}{-4}$

$(x, y) = (-4, -\frac{11}{4})$

$y - y_1 = m(x - x_1)$

$\boxed{y + \frac{11}{4} = \frac{16}{11}(x + 4)}$

---

9. Determine all of the *x* values of the function $f(x) = (1/2)x^2 + 5x$ at which tangent lines are horizontal.

$f'(x) = (\frac{1}{2})2x + 5 = 0 \leftarrow$ slope of horiz tan line

$x + 5 = 0$

$\boxed{x = -5}$

---

10. If $s(t) = t^2 - 6t$ feet is the time-position function (with *t* given in seconds), what is the velocity function?

$v(t) = s'(t) = \boxed{(2t - 6)\text{ ft/sec}}$

11. What are all of the **numerical** *x* values of the function $f(x) = x^3 - 3x^2$ at which tangent lines have a slope of $\sqrt{2}$?

$$m = f'(x) = 3x^2 - 6x = \sqrt{2}$$

$$3x^2 - 6x - \sqrt{2} = 0$$

$$a = 3, \; b = -6, \; c = -\sqrt{2}$$

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

$$x = \frac{6 \pm \sqrt{36 - 4(3)(-\sqrt{2})}}{2 \cdot 3} \rightarrow x = \frac{6 \pm \sqrt{36 + 12\sqrt{2}}}{6}$$

$$\boxed{\begin{aligned} x &= 2.213 \\ x &= -.213 \end{aligned}}$$

---

12. What is (are) the *x* position(s) on the curve given by $y = x^2 - 10x + 9$ at which normal lines are exactly vertical?

$$m_{tan} = y' = 2x - 10 = 0 \leftarrow$$ If tan line is horiz, then normal line will be vert.

$$2x = 10$$

$$\boxed{x = 5}$$

---

13. What is the instantaneous rate of change of $f(x) = x^4 - x + 1$ at $x = 2$?

$$f'(x) = 4x^3 - 1$$

$$f'(2) = 4 \cdot 2^3 - 1$$

$$= 4 \cdot 8 - 1$$

$$= \boxed{31}$$

14. Find $h'(-1)$ where $h(t) = t^5 + 6t$.

$$h'(t) = 5t^4 + 6$$

$$h'(-1) = 5(-1)^4 + 6$$

$$= 5 \cdot 1 + 6$$

$$= \boxed{11}$$

Unit 3: Lesson 02 Product and quotient rules

Product rule:

If f(x) = u(x) v(x), then

f'(x) = u · v' + v · u'

Example 1: Find the derivative of $f(x) = \sqrt[3]{x}(x^2 + 5)$.

$$f(x) = \sqrt[3]{x}\,(x^2+5) = \underset{u}{x^{1/3}}\underset{v}{(x^2+5)}$$

$$f' = uv' + vu'$$

$$= \boxed{x^{1/3}(2x) + (x^2+5)\left(\tfrac{1}{3}x^{-2/3}\right)}$$

Quotient rule:

$$\text{If } f(x) = \frac{u}{v}$$

$$\mathbf{f'(x) = \frac{v \cdot u' - u \cdot v'}{v^2}}$$

See **Enrichment topic D** for verification of both the product and quotient rules.

Example 2: Find the derivative of $f(x) = \dfrac{\sqrt{x}}{x + 3x^4}$

$$f(x) = \frac{\sqrt{x}}{x+3x^4} = \frac{x^{1/2}}{x+3x^4}$$

$$f'(x) = \boxed{\frac{(x+3x^4)\left(\frac{1}{2}x^{-1/2}\right) - x^{\frac{1}{2}}(1+12x^3)}{(x+3x^4)^2}} = \frac{vu' - uv'}{v^2}$$

Assignment: In problems 1-8, find the derivatives of the given functions.

1. $f(x) = (5x - 11)/(2x - 1)$

$$f(x) = \frac{5x-11 \leftarrow u}{2x-1 \leftarrow v}$$

$$f' = \frac{(2x-1)5 - (5x-11)2}{(2x-1)^2}$$

$$f' = \frac{\cancel{10x} - 5 - \cancel{10x} + 22}{(2x-1)^2}$$

$$f' = \boxed{\frac{17}{(2x-1)^2}}$$

2. $7x(\sqrt{x})$

$$f(x) = \underset{u}{7x}\underset{v}{\sqrt{x}} = 7x(x^{1/2})$$

$$f' = uv' + vu'$$

$$= 7x(\tfrac{1}{2}x^{-1/2}) + x^{1/2}7$$

$$= \frac{7}{2}x^{1/2} + 7x^{1/2}\frac{2}{2}$$

$$= \boxed{\frac{21}{2}x^{1/2}}$$

3. $g(x) = (x + 6)/\sqrt{x}$

$$g(x) = \frac{x+6 \leftarrow u}{x^{1/2} \leftarrow v} \qquad g' = \frac{vu' - uv'}{v^2}$$

$$g' = \frac{x^{1/2}(1) - (x+6)(\frac{1}{2}x^{-1/2})}{(x^{1/2})^2} = \frac{x^{1/2} - \frac{1}{2}x^{1/2} - 3x^{-1/2}}{x}$$

$$= \frac{\frac{1}{2}x^{1/2} - 3x^{-1/2}}{x}\;\frac{x^{1/2}}{x^{1/2}} = \frac{\frac{1}{2}x - 3}{x^{3/2}}\;\frac{2x^{1/2}}{2x^{1/2}}$$

$$= \boxed{\frac{x^{3/2} - 6x^{1/2}}{2x^2}}$$

4. $L(w) = 7w/(8w^2 + 2)$

$$L = \frac{7w \leftarrow u}{8w^2+2 \leftarrow v} \qquad L' = \frac{vu' - uv'}{v^2}$$

$$L' = \frac{(8w^2+2)7 - 7w(16w)}{(8w^2+2)^2} = \frac{56w^2 + 14 - 112w^2}{(8w^2+2)^2}$$

$$L' = \boxed{\frac{-56w^2 + 14}{(8w^2+2)^2}}$$

5. $h(p) = (p – 1)/(p^2 + 4p – 8)$

$$h = \frac{p-1}{p^2+4p-8} \begin{matrix} \leftarrow u \\ \leftarrow v \end{matrix}$$

$$h' = \frac{(p^2+4p-8)1-(p-1)(2p+4)}{(p^2+4p-8)^2}$$

$$h' = \frac{p^2+4p-8-2p^2-2p+4}{(p^2+4p-8)^2}$$

$$h' = \boxed{\frac{-p^2+2p-4}{(p^2+4p-8)^2}}$$

6. $f(x) = (–x^3 + 12x^2 – 4)7$

$$= -7x^3+84x^2-28$$

$$f' = \boxed{-21x^2+168x}$$

7. $y = (t + 7)(t^7 – 8t^2 + t – 6)$

$$y = \underset{u}{(t+7)}\underset{v}{(t^7-8t^2+t-6)}$$

$$y' = u\,v' + v\,u'$$

$$y' = (t+7)(7t^6-16t+1) + (t^7-8t^2+t-6)(1)$$

$$y' = 7t^7-16t^2+49t^6-111t+7 + t^7-8t^2+t-6$$

$$\boxed{y' = 8t^7+49t^6-24t^2-110t+1}$$

8. $f(x) = (x – 7)\ 59\ \sqrt[5]{x}$

$$f(x) = \underset{u}{(x-7)}\underset{v}{(59x^{1/5})}$$

$$f' = (x-7)\frac{59}{5}x^{-4/5} + 59x^{1/5}(1)$$

$$f' = \frac{59(x-7)}{5x^{4/5}}\frac{x^{1/5}}{x^{1/5}} + 59x^{1/5}$$

$$f' = \boxed{59x^{1/5}\left(\frac{(x-7)}{5x} + 1\right)}$$

9. Find the instantaneous rate of change of $f(x) = \sqrt{x}\ \sqrt[5]{x}$ at x = 5.

$$f(x) = \underset{u}{x^{1/2}}\ \underset{v}{x^{1/5}}$$

$$f'(x) = x^{1/2}\left(\frac{1}{5}x^{-4/5}\right) + x^{1/5}\left(\frac{1}{2}x^{-1/2}\right)$$

$$= \frac{1}{5}x^{-3/10} + \frac{1}{2}x^{-3/10}$$

$$= \frac{7}{10}x^{-3/10} = \frac{7}{10x^{3/10}}\frac{x^{7/10}}{x^{7/10}} = \frac{7x^{7/10}}{10x} = \frac{7(5)^{7/10}}{10(5)}$$

$$= \frac{7\cdot 5^{7/10}}{50} = \boxed{.4319237}$$

10. What is the velocity as a function of time (in seconds) of the time-position function s(t) = (t^3 – 2t)/t^5 meters?

$$s(t) = \frac{t^3-2t \leftarrow u}{t^5 \leftarrow v}$$

$$v(t) = s'(t) = \frac{t^5(3t^2-2)-(t^3-2t)(5t^4)}{(t^5)^2}$$

$$= \frac{3t^7-2t^5-5t^7+10t^5}{t^{10}} = \frac{-2t^7+8t^5}{t^{10}}$$

$$= \frac{\cancel{t^5}(-2t^2+8)}{t^{\cancel{10}5}} = \boxed{\frac{-2t^2+8}{t^5}\ m/sec}$$

---

11. Evaluate dy/dx at x = 2 where y = x/$\sqrt{8x}$.

$$y = \frac{x}{(8x)^{1/2}} = \frac{x}{8^{1/2}x^{1/2}} = \frac{x}{2\sqrt{2}\,x^{1/2}} = \frac{x^{1/2}}{2\sqrt{2}}$$

$$y' = \frac{\frac{1}{2}x^{-1/2}}{2\sqrt{2}} = \frac{1}{4\sqrt{2}x^{1/2}}\,\frac{x^{1/2}\sqrt{2}}{x^{1/2}\sqrt{2}}$$

$$= \frac{\sqrt{2}x^{1/2}}{4\cdot 2x} = \frac{\sqrt{2}\,x^{1/2}}{8x}$$

$$y'(2) = \frac{\sqrt{2}\sqrt{2}}{8(2)} = \frac{2}{16} = \boxed{\frac{1}{8}}$$

---

12. Find the tangent line to f(x) = (4x – 6)x$^{2/3}$ at x = 8.

$$f(x) = (\underset{u}{4x-6})\underset{v}{x^{2/3}}$$

$$m = f' = (4x-6)(2/3\,x^{-1/3}) + x^{2/3}(4)$$

$$m = f'(8) = (4\cdot 8-6)\left(\frac{2}{3\cdot 8^{1/3}}\right) + (8^{1/3})^2(4) = 26\left(\frac{4}{3}\right) + 16 = \frac{74}{3}$$

$$f(8) = (4\cdot 8-6)(8^{1/3})^2 = 26(4)$$

$$= 104$$

$$(x, y) = (8, 104)$$

$$y - y_1 = -m(x - x1)$$

$$\boxed{y - 104 = \frac{24}{3}(x-8)}$$

13. Find the y-intercept at x = 1 of the normal line to the curve given by f(x) = $\frac{\left(x+\frac{2}{3}\right)}{5\sqrt{x}}$.

$$m_{tan} = f' = \frac{5\sqrt{x}(1) - \left(x+\frac{2}{3}\right)\left(5\left(\frac{1}{2}\right)x^{-1/2}\right)}{(5\sqrt{x})^2}$$

$$f'(1) = \frac{5\sqrt{1} - \left(\frac{3}{3}+\frac{2}{3}\right)\left(\frac{5}{2}(1)^{-1/2}\right)}{(5\sqrt{1})^2}$$

$$= \frac{5 - \frac{5}{3}\left(\frac{5}{2}\right)}{25} = \frac{5-\frac{25}{6}}{25} = \frac{\frac{5}{6}}{25} = \frac{5}{150} = \frac{1}{30}$$

$$m_{\perp} = -30$$

$$f(1) = \frac{1+3/3}{5\sqrt{1}} = \frac{5/3}{5}$$

$$= \frac{1}{3}$$

$$(x_1, y_1) = \left(1, \frac{1}{3}\right)$$

$y = mx+b$

$y = -30x+b$ sub in $\left(1, \frac{1}{3}\right)$

$\frac{1}{3} = -30(1)+b$

$\frac{1}{3} + \frac{30}{1} = b$

$\frac{1}{3} + \frac{30}{1}\frac{3}{3} = b$

$$\boxed{\frac{91}{3} = b}$$

Unit 3: Lesson 03

Trig function derivatives

Trig derivatives:

$$\frac{d}{dx}\sin(x) = \cos(x) \qquad \frac{d}{dx}\csc(x) = -\csc(x)\cot(x)$$

$$\frac{d}{dx}\cos(x) = -\sin(x) \qquad \frac{d}{dx}\sec(x) = \sec(x)\tan(x)$$

$$\frac{d}{dx}\tan(x) = \sec^2(x) \qquad \frac{d}{dx}\cot(x) = -\csc^2(x)$$

See **Enrichment topic E** for a derivation of the rules for sine and cosine.

Example 1: If $f(x) = x^3\sin(x)$ find $f'(x)$.

$u = x^3 \quad v = \sin(x)$

$f' = uv' + vu'$

$= \boxed{x^3\cos(x) + \sin(x)(3x^2)}$

Example 2: If $f(x) = \sin(x)\sec(x)$ find f'.

$u = \sin(x) \quad v = \sec(x)$

$f' = uv' + vu'$

$f' = \cancel{\sin(x)}(\overset{\tan(x)}{\cancel{\sec(x)}}\tan(x)) + \underset{1}{\cancel{\sec x \cos x}}$

$= \tan^2 x + 1$

$= \boxed{\sec^2 x}$

Example 3: Using the identity $\tan(x) = \sin(x)/\cos(x)$ show that the derivative of $\tan(x)$ is $\sec^2(x)$.

$\tan(x) = \dfrac{\sin(x)}{\cos(x)} \begin{matrix} \leftarrow u \\ \leftarrow v \end{matrix}$; $f' = \dfrac{vu' - uv'}{v^2}$

$f' = \dfrac{\cos(x)\cos(x) - \sin(x)(-\sin(x))}{\cos^2(x)}$

$= \dfrac{\overset{1}{\cancel{\cos^2(x) + \sin^2(x)}}}{\cos^2(x)} = \dfrac{1}{\cos^2(x)} = \left(\dfrac{1}{\cos(x)}\right)^2 = \boxed{\sec^2(x)}$

Assignment: In problems 1-10, find the derivative of the given function.

1. f(x) = x sin(x)

$u = x ; v = \sin(x)$

$f' = uv' + vu'$

$= x\cos(x) + \sin(x)\cdot 1$

$= \boxed{x\cos(x) + \sin(x)}$

2. f(x) = $(x^2 + 1)$tan(x)

$u = x^2+1 ; v = \tan x$

$f' = uv' + vu'$

$f' = \boxed{(x^2+1)\sec^2(x) + \tan(x)(2x)}$

3. f(x) = x^2/sec(x)

$u = x^2 ; v = \sec(x)$

$f' = \frac{vu' - uv'}{v^2}$

$= \boxed{\frac{\sec(x)(2x) - x^2\sec(x)\tan(x)}{\sec^2(x)}}$

4. f(x) = csc(x)/$(x^3 - 8)$

$u = \csc(x) ; v = x^3 - 8$

$f' = \frac{vu' - uv'}{v^2}$

$= \boxed{\frac{(x^3-8)(-\csc(x)\cot(x)) - \csc(x)3x^2}{(x^3-8)^2}}$

5. g(t) = sin(t) cos(t)

$u = \sin(t); v = \cos(t)$

$g' = uv' + vu'$

$= \sin(t)(-\sin(t)) + \cos(t)\cos(t)$

$= \boxed{-\sin^2(t) + \cos^2(t)}$

6. P(θ) = 7tan(θ)

$P' = \boxed{7\sec^2(\theta)}$

7. $f(x) = \dfrac{\cot(x) + 9}{\sqrt{x}}$

$u = \cot(x) + 9 \; ; \; v = x^{1/2}$

$$f' = \frac{vu' - uv'}{v^2}$$

$$= \frac{x^{1/2}(-\csc^2 x) - (\cot x + 9)\frac{1}{2}x^{-1/2}}{(x^{1/2})^2}$$

$$= \frac{-\sqrt{x}\csc^2(x) - (\cot x + 9)\frac{1}{2\sqrt{x}}\frac{\sqrt{x}}{\sqrt{x}}}{x}$$

$$= \frac{-\sqrt{x}\csc^2 x - (\cot x + 9)\frac{\sqrt{x}}{2x}}{x}\frac{2x}{2x}$$

$$= \boxed{\frac{-2x^{3/2}\csc^2(x) - (\cot x + 9)\sqrt{x}}{2x^2}}$$

8. $f(x) = \dfrac{\sin(x)}{\cos(x) + 2}$

$u = \sin(x) \; ; \; v = \cos(x) + 2$

$$f' = \frac{vu' - uv'}{v^2}$$

$$= \frac{(\cos(x) + 2)\cos(x) - \sin(x)(-\sin(x))}{(\cos(x) + 2)^2}$$

$$= \frac{\overbrace{\cos^2(x) + 2\cos(x) + \sin^2 x}^{1}}{(\cos(x) + 2)^2}$$

$$= \boxed{\frac{1 + 2\cos(x)}{(\cos(x) + 2)^2}}$$

9. y =t^2(sin(t) + cot(t))

$u = t^2 \; ; \; v = \sin(t) + \cos(t)$

$y' = uv' + vu'$

$= \boxed{t^2(\cos(t) - \csc^2 t) + (\sin t + \cot(t))2t}$

10. y = tan(x) cot(x)

$y = \tan(x)\cot(x) = 1$

$\boxed{y' = 0}$

11. Using an identity for csc(x), show that its derivative is – csc(x) cot(x).

$y = \csc(x) = \dfrac{1}{\sin(x)} \; ; \; u = 1 \; ; \; v = \sin(x)$

$$y' = \frac{\sin(x)(0) - 1\cos(x)}{\sin^2(x)} = \frac{-1}{\sin(x)}\,\frac{\cos(x)}{\sin(x)}$$

$$= \boxed{-\csc(x)\cot(x)}$$

12. Develop the rule for the derivative of cot(x).

$y = \cot(x) = \dfrac{\cos(x)}{\sin(x)} \qquad u = \cos(x)\ ;\ v = \sin(x)$

$y' = \dfrac{vu' - uv'}{v^2}$

$= \dfrac{\sin(x)(-\sin(x)) - \cos(x)\cos(x)}{\sin^2 x}$

$= \dfrac{\cancel{-\sin^2(x) - \cos^2(x)} \rightarrow -1}{\sin^2 x}$

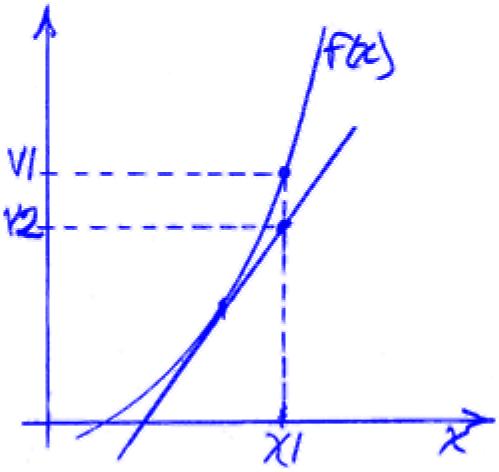
$= \dfrac{-1}{\sin^2 x} = -\left(\dfrac{1}{\sin(x)}\right)^2 = \boxed{-\csc^2(x)}$

Unit 3: Lesson 04

Linear approximations
Derivatives of piecewise functions

The **tangent line** to a curve can be used to obtain an approximation to function values of the curve **near** the point of tangency.

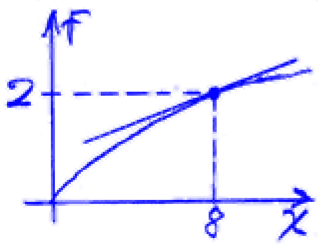
In the adjacent drawing, the true value of the function at *x1* is *v1*. Note that *v2*, the **approximate** value of *f*, is actually the value of the linear function at *x1*.



Notice in the drawing above that the estimate (V2) for *f* is l**ow** because the tangent line is **below** the curve.

Had the tangent line been **above** the curve, the estimate would have been **high**.

Example 1: What is a linear approximation to the curve $f(x) = \sqrt[3]{x}$ at x = 8.01? Is this estimate higher or lower than the true value? Why?



Use point $(8,2) = (x_1, y_1)$

$$f' = \frac{1}{3}x^{-2/3}$$

$$f'(8) = \frac{1}{3}(8^{1/3})^{-2} = \frac{1}{3}\cdot 2^{-2} = \frac{1}{12}$$

$$y - y_1 = m(x - x_1)$$

$$y - 2 = \frac{1}{12}(x - 8)$$

$$y = \frac{1}{12}(x-8) + 2$$

$$y(8.01) = \frac{1}{12}(8.01-8) + 2$$

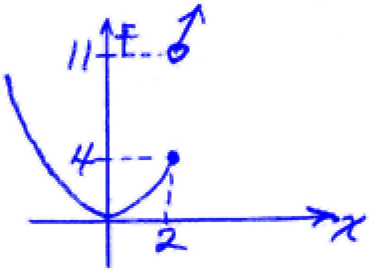
$$= \boxed{2.00083}$$

Estimate is high
Tangent line is above the curve.

Piecewise functions will naturally result in the derivative also being piecewise.

Example 2: Find the derivative of the following piecewise function:

$$f(x) = \left\{\begin{matrix} x^2 & \text{if } x \le 2 \\ 4x + 3 & \text{if } x > 2 \end{matrix}\right\}$$

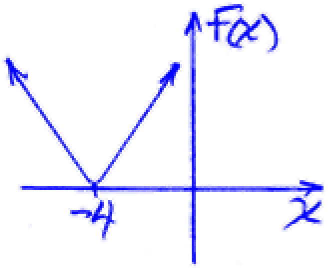


$$f'(x) = \begin{cases} 2x & \text{if } x < 2 \\ 4 & \text{if } x > 2 \end{cases}$$

Absolute value functions are easily represented as piecewise functions.

When asked to take the derivative of an absolute value function, first **convert it to piecewise form**.

Example 3: Find the derivative of f(x) = |x + 4|



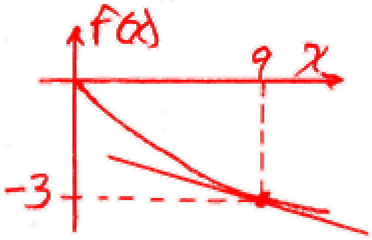
$$f(x) = \begin{cases} -x-4 & \text{if } x \le -4 \\ x+4 & \text{if } x > -4 \end{cases}$$

$$\boxed{f'(x) = \begin{cases} -1 & \text{if } x < -4 \\ 1 & \text{if } x > -4 \end{cases}}$$

cusp at $x=-4$; Der is not defined there

Assignment:

1. Find a linear approximation to the curve $f(x) = -x^{1/2}$ at x = 8.98. Is this estimate higher or lower than the true value? Why?



Estimate is low. Tan line is below the curve.

$f'(x) = -\frac{1}{2}x^{-\frac{1}{2}}$

$m = f'(9) = -\frac{1}{2}(9)^{-\frac{1}{2}} = -\frac{1}{2\cdot 3} = -\frac{1}{6}$

$(x_1, y_1) = (9, -3)$

$y - y_1 = m(x - x_1)$

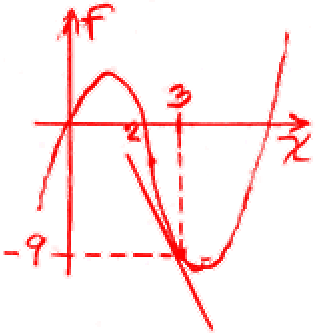
$y + 3 = -\frac{1}{6}(x-9)$; $y = -\frac{1}{6}(x-9) - 3$

$y(8.98) = -\frac{1}{6}(8.98 - 9) - 3$

$= \boxed{-2.99\overline{6}}$

---

2. Find a linear approximation to the curve $f(x) = x^3 - 8x^2 + 12x$ at x = 2.9. Is this estimate higher or lower than the true value? Why?



$f'(x) = 3x^2 - 16x + 12$

$f'(3) = 3(3)^2 - 16(3) + 12 = -9 = m$

$(x_1, y_1) = (3, -9)$

$y - y_1 = m(x - x_1)$; $y + 9 = -9(x - 3)$

$y = -9(x-3) - 9$

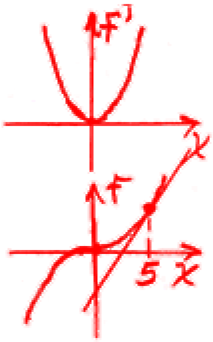
$y(2.9) = -9(2.9 - 3) - 9$

$= \boxed{-8.1}$

Estimate is low. Tangent line is below the curve.

---

3. Suppose we know that the derivative of a function to be f'(x) = $2x^2$ and that f(5) = 4. What is a linear approximation for the function value at x = 5.06? Is this estimate higher or lower than the true value? Why?



$f(5) = +4 \rightarrow (x_1, y_1) = (5, 4)$

$m = f'(5) = 2\cdot 5^2 = 50$

$y - y_1 = m(x - x_1)$

$y - 4 = 50(x - 5)$

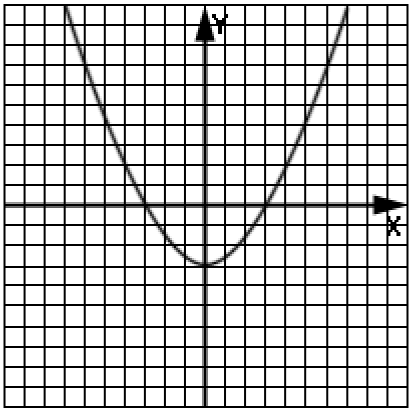
$y = 50(x-5) + 4$

$y(5.06) = 50(5.06 - 5) + 4$

$= \boxed{7}$

Estimate is low. Tangent line is below curve.

4. The adjacent graph shows f′(x). Find a linear approximation of f(−3.98) when f(−4) = 5. Is this estimate higher or lower than the true value? Why?



From the drawing $m = f'(-4) = 2$

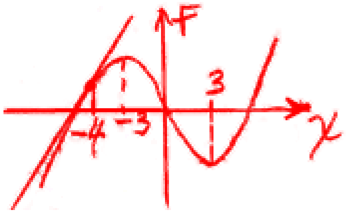
$f(-4) = 5 \rightsquigarrow (-4, 5)$

$y - y_1 = m(x - x_1)$; $y - 5 = 2(x+4)$

$y = 2(x+4) + 5$

$y(-3.98) = 2(-3.98+4)+5 = \boxed{5.04}$

Estimate is high,
Tangent line is above
the curve



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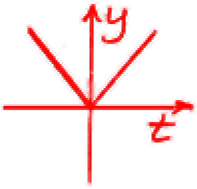
5. Find the derivative of the following piecewise function:

$$f(x) = \begin{cases} \sin(x) & \text{if } x \geq \frac{\pi}{2} \\ \tan(x) & \text{if } x < \frac{\pi}{2} \end{cases}$$

$$f'(x) = \begin{cases} \cos(x) & \text{if } x > \pi/2 \\ \sec^2(x) & \text{if } x < \pi/2 \end{cases}$$

---

6. What is the derivative of y = |t|?



$$y = \begin{cases} x & \text{if } x \geq 0 \\ -x & \text{if } x < 0 \end{cases}$$

$$\boxed{y' = \begin{cases} 1 & \text{if } x > 0 \\ -1 & \text{if } x < 0 \end{cases}}$$

7. Find the derivative of a piecewise function that is defined by $f(x) = x^3 + 1$ to the left of x = – 6 and by f(x) = 6 at x = –6 and to the right of x = – 6.

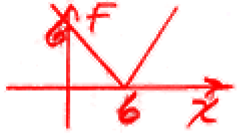
$$f(x)=\begin{cases} x^3+1 & \text{if } x<-6 \\ 6 & \text{if } x\ge -6 \end{cases}$$

$$\boxed{f'(x)=\begin{cases} 3x^2 & \text{if } x<-6 \\ 0 & \text{if } x>-6 \end{cases}}$$

---

8. What is the derivative of f(x) = |.5x – 3|?

$$\tfrac{1}{2}x-3=0$$
$$\tfrac{1}{2}x=3$$
$$x=6$$



$$f(x)=\begin{cases} x-6 & x\ge 6 \\ -x+6 & x<6 \end{cases}$$

$$\boxed{f'(x)=\begin{cases} 1 & x>6 \\ -1 & x<6 \end{cases}}$$

---

9. If f(x) = g(x) h(x) find f'(5) when g(5) = 3, g'(5) = –1, h(5) = 22, and h'(5) = 4.

$$f'(x)=gh'+hg'$$
$$f'(5)=g(5)h'(5)+h(5)g'(5)$$
$$=3(4)+22(-1)$$
$$=12-22=\boxed{-10}$$

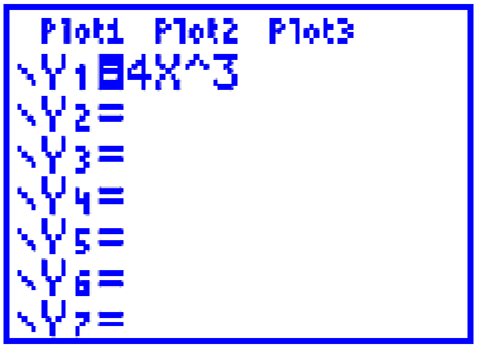
Unit 3: Lesson 05

Derivatives on the graphing calculator

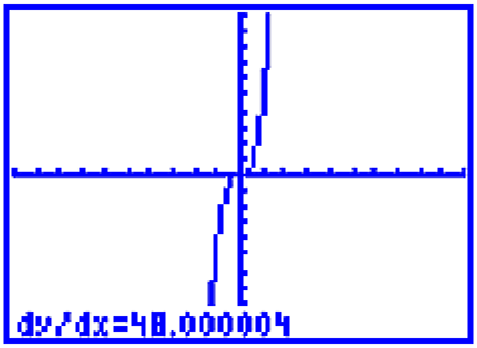
See **Calculator Appendix AF** for two techniques for finding the derivative of a function evaluated at a particular point.

The second technique, using **MATH | 8: nDeriv(**, is generally the best and least troublesome.

Example 1: Use a calculator to find the derivative of f(x) = $4x^3$ at x =2. Confirm the calculator answer with a "hand" calculation.



```
 Plot1  Plot2  Plot3
\Y1=4X^3
\Y2=
\Y3=
\Y4=
\Y5=
\Y6=
\Y7=
```



```
dy/dx=48.000004
```

$f'(x) = 12x^2$

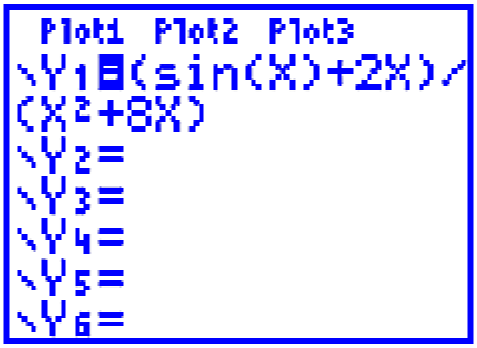
$f'(2) = 12 \cdot 2^2$

$= 12 \cdot 4$

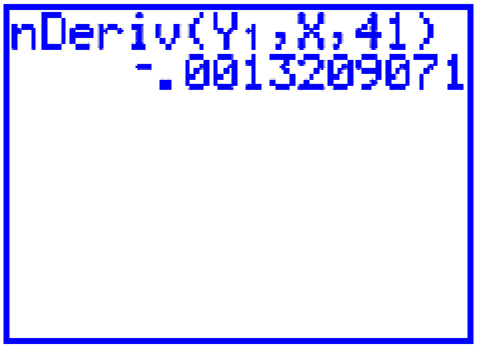
$= \boxed{48}$

Not exact due to round-off error

Example 2: Use a calculator to find the derivative of f(x) = $(\sin(x) + 2x)/(x^2 + 8x)$ evaluated at x = 41. (Assume *x* is in radians.)



```
 Plot1  Plot2  Plot3
\Y1=(sin(X)+2X)/
(X²+8X)
\Y2=
\Y3=
\Y4=
\Y5=
\Y6=
```



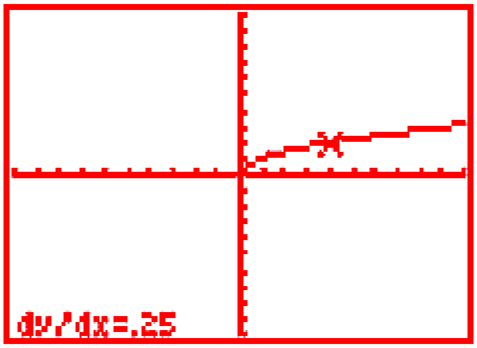
```
nDeriv(Y1,X,41)
       -.0013209071
```

Teachers: Since this is a relatively short lesson, it is suggested that the students also begin work on the cumulative review after finishing this assignment. . . or, better still, present Enrichment topic C or D.

Assignment:

1. Use a calculator to find the derivative of $f(x) = \sqrt{x}$ evaluated at x = 4. Confirm the calculator answer with a "hand" calculation.

```
 Plot1  Plot2  Plot3
\Y1=√(X)
\Y2=
\Y3=
\Y4=
\Y5=
\Y6=
\Y7=
```



```
dy/dx=.25
```

$f(x) = x^{1/2}$

$f'(x) = \frac{1}{2}x^{-\frac{1}{2}}$

$f'(4) = \frac{1}{2\cdot 4^{1/2}}$

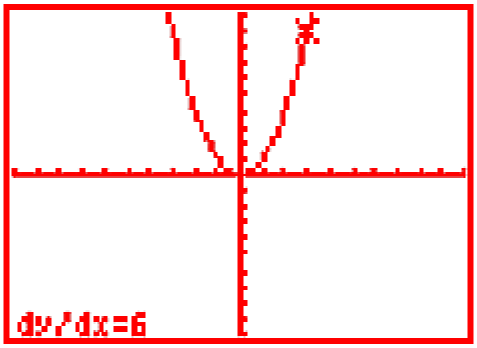
$= \frac{1}{2\cdot 2}$

$= \boxed{\frac{1}{4}}$

---

2. Use a calculator to find the derivative of $f(x) = x^2$ evaluated at x = 3. Confirm the calculator answer with a "hand" calculation.

```
 Plot1  Plot2  Plot3
\Y1=X²
\Y2=
\Y3=
\Y4=
\Y5=
\Y6=
\Y7=
```



```
dy/dx=6
```

$f'(x) = 2x$

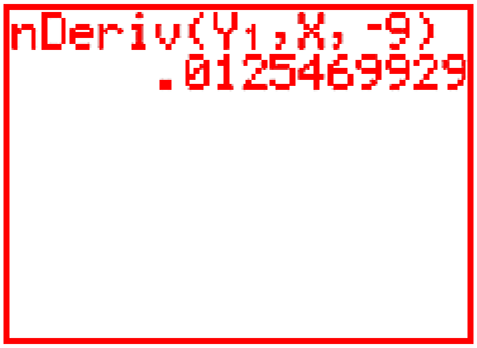
$f'(3) = 2\cdot 3$

$= \boxed{6}$

---

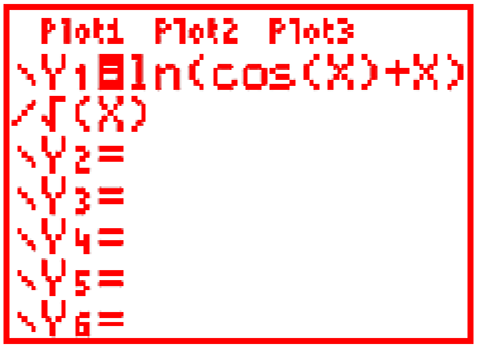
3. Use a calculator to find the derivative of $f(x) = \sqrt{\tan(x)}\,/(x^2 + 2)$ evaluated at x = –9. (Assume *x* is in radians.)

```
 Plot1  Plot2  Plot3
\Y1=√(tan(X))/(X
²+2)
\Y2=
\Y3=
\Y4=
\Y5=
\Y6=
```

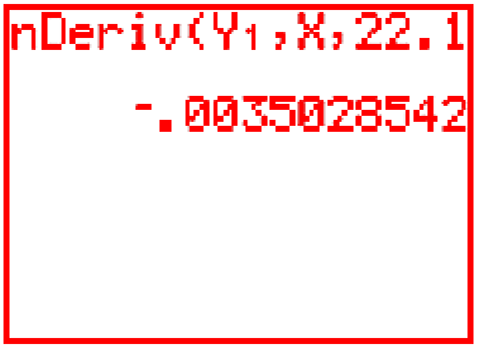


```
nDeriv(Y1,X,-9)
         .0125469929
```

4. Use a calculator to find the derivative of $y = \ln(\cos(x) + x)/\sqrt{x}$ at x = 22.1. (Assume *x* is in radians.)



```
  Plot1  Plot2  Plot3
\Y1=ln(cos(X)+X)
/√(X)
\Y2=
\Y3=
\Y4=
\Y5=
\Y6=
```



```
nDeriv(Y1,X,22.1
      -.0035028542
```

Unit 3:
Cumulative Review

1. $\lim\limits_{h\to 0}\dfrac{\tan(x+h)-\tan(x)}{h} = ?$ at $x =$ π radians.

A. 1 B. 0 C. $\sqrt{3}/2$

D. 1/2 E. None of these

$$\lim_{h\to 0}\frac{\tan(x+h)-\tan(x)}{h} = \frac{d}{dx}\tan(x) = f'(x)$$

$$= \sec^2(x)$$

$$f'(\pi) = \sec^2(\pi) = (-1)^2 = \boxed{1}$$

---

2. $\lim\limits_{h\to 0}\dfrac{3(x+h)^2-8(x+h)-5-3x^2+8x+5}{h} = ?$

A. $3x^2-8x-5$ **B. $6x-8$** C. 0

D. Undefined E. None of these

$$f(x) = 3x^2-8x-5$$

$$f'(x) = \boxed{6x-8}$$

3. $\lim_{h\to 0}\dfrac{\cos(\pi+h)-\cos(\pi)}{h}=?$

A. cos(x)

B. sin(x)

C. −sin(x)

D. 0

E. None of these

This could be worked strictly as a limit problem as shown below… or it could be noticed that it's really the derivative of cos(x) evaluated at x = π… -sin(π) = 0.

$$= \lim_{h\to 0}\frac{\cos(\pi+h)-\cos(\pi)}{h}$$

$\cos(A+B)=\cos A\cos B$
$-\sin A\sin B$

$$= \lim_{h\to 0}\frac{\overset{-1}{\cos(\pi)}\cos(h)-\overset{0}{\sin(\pi)}\overset{\approx h}{\sin(h)}-\overset{-1}{\cos(\pi)}}{h}$$

$$= \lim_{h\to 0}\frac{-1\cos(h)-0+1}{h} = \lim_{h\to 0}\frac{1-\cos(h)}{h} = \boxed{0}$$

---

4. State the problem posed by this table in (one-sided) limit notation along with what it seems to be approaching.

x	f(x)
–4.12	1/10
–4.11	–1/100
–4.103	1/1000
–4.10054	–1/100,000
–4.100003	1/1,000,000

A. $\lim f(x) = \infty$

B. $\lim_{x\to -4^+} f(x) = 0$

C. $\lim_{x\to 4^-} f(x) = \infty$

D. $\lim_{x\to -4.1^+} f(x) = \infty$

E. $\lim_{x\to -4.1^-} f(x) = 0$

F. None of these

5. $\lim\limits_{x \to 16} \dfrac{16 - x}{\sqrt{x} - 4} = ?$

A. 4
B. -4
C. 0
D. +∞
E. None of these

$$\lim_{x\to 16} \frac{16-x}{\sqrt{x}-4} \cdot \frac{\sqrt{x}+4}{\sqrt{x}+4} = \lim_{x\to 16} \frac{\overset{-1}{\cancel{(16-x)}}(\sqrt{x}+4)}{\cancel{x-16}}$$

$$= \lim_{x\to 16} -(\sqrt{x}+4) = -(\sqrt{16}+4) = -(4+4) = \boxed{-8}$$

---

6. $\lim\limits_{x \to 0} \dfrac{\sin(2x)}{x} = ?$

A. 2x
B. x/2
C. 1/2
D. 2
E. None of these

$$\lim_{x\to 0} \frac{2\sin(2x)}{2x} = 2\lim_{x\to 0} \frac{\sin(2x)}{2x} = 2 \cdot 1 = \boxed{2}$$

---

7. What is the average rate of change of f(x) = x^3 – x between x = 0 and x = 3?

A. 9
B. $3x^2 - 1$
C. 8
D. (f(3) – f(0))/3
E. More than one of these

$$\frac{f(3)-f(0)}{3} = \frac{3^3-3-(0^3-0)}{3} = \frac{27-3-0}{3}$$

$$= \frac{24}{3} = \boxed{8}$$

C + D are both correct

8. What is the instantaneous rate of change of $f(x) = x^3 - x$ at $x = 3$?

A. $3x^2 - 1$ B. $f'(3)$ C. $f(3)$

D. 26 **E. More than one of these**

$f'(x) = 3x^2 - 1$

$f'(3) = 3 \cdot 3^2 - 1$

$= 27 - 1$

$= \boxed{26}$

B + D are both correct

---

9. What is the equation of the normal line to the curve $1/x$ at $x = 2$?

A. $y - 5 = -.25(x - 2)$ B. $y = 4x - 7/2$ C. $y = -.25x + 1$

D. $y = 4x - 15/2$ E. More than one of these

$f(x) = x^{-1}$

$f'(x) = -1x^{-2}$

$f'(2) = -1 \cdot 2^{-2}$

$m_{tan} = -\frac{1}{4}$

$f(2) = \frac{1}{2}$

$(x_1, y_1) = (2, \frac{1}{2})$

$m_{\perp} = 4$

$y - y_1 = m(x - x_1)$

$y - \frac{1}{2} = 4(x - 2)$

$y = 4x - 8 + \frac{1}{2}$

$\boxed{y = 4x - \frac{15}{2}}$

10. What is the velocity of an object whose time-position function is given by s(t) = (5/4)t^2 – 6t meters at t = 6 sec?

A. 9 meters

B. s'(6) m/sec

C. 15 sec

D. 6s'(t) m/sec

E. None of these

$$s'(t) = v(t) = \frac{5}{4}(2)t - 6$$

$$= \frac{5}{2}t - 6$$

$$s'(6) = v(6) = \frac{5}{2}(6) - 6$$

$$= \frac{30}{2} - 6$$

$$= 15 - 6$$

$$= \boxed{9 \text{ m/sec}}$$

Unit 3: Review

In problems 1-8, find the derivatives of the given functions.

1. f(x) = $2x^4 - 6x + 11$

$$f'(x) = \boxed{8x^3 - 6}$$

2. f(x) = $(x^2 + x)/x$

$$F(x) = \frac{\cancel{x}(x+1)}{\cancel{x}}$$

$$= x+1$$

$$f'(x) = \boxed{1}$$

3. $f(x) = 6\sqrt{x}$

$$F(x) = 6x^{\frac{1}{2}}$$

$$f'(x) = 6\left(\tfrac{1}{2}\right)x^{-1/2}$$

$$= \frac{3}{x^{1/2}} \quad \frac{x^{1/2}}{x^{1/2}}$$

$$= \frac{3x^{1/2}}{x} = \boxed{\frac{3\sqrt{x}}{x}}$$

4. $g(t) = t^3(\sqrt{t} + t)$

$$u = t^3 \quad v = (\sqrt{t} + t)$$

$$g' = uv' + vu'$$

$$g' = t^3\left(\tfrac{1}{2}t^{-1/2} + 1\right) + (\sqrt{t} + t)\,3t^2$$

$$g' = \tfrac{1}{2}t^{5/2} + t^3 + 3t^{5/2} + 3t^3$$

$$= \boxed{\tfrac{7}{2}t^{5/2} + 4t^3}$$

5. $P(q) = (q^3 + 4q)/(q - \sqrt[3]{q})$

$$u = q^3 + 4q \qquad v = q - q^{1/3}$$

$$P' = \frac{vu' - uv'}{v^2} \qquad = \frac{(q - q^{1/3})(3q^2 + 4) - (q^3 + 4q)(1 - \frac{1}{3}q^{-2/3})}{(q - q^{1/3})^2}$$

$$= \frac{3q^3 - 3q^{7/3} + \cancel{4q} - 4q^{1/3} - q^3 \cancel{-4q} + \frac{1}{3}q^{7/3} + \frac{4}{3}q^{1/3}}{(q - q^{1/3})^2}$$

$$= \boxed{\frac{2q^3 - \frac{8}{3}q^{7/3} - \frac{8}{3}q^{1/3}}{(q - q^{1/3})^2}}$$

6. f(θ) = sin(θ) (tan(θ) + 1)

$u = \sin\theta \quad v = \tan\theta + 1$

$f' = uv' + vu'$

$= \sin\theta(\sec^2\theta) + (\tan\theta + 1)\cos\theta$

$= \boxed{\sin\theta\sec^2\theta + \sin\theta + \cos\theta}$

7. f(t) = (t^2+6t)/(t + 1)

$u = t^2 + 6t \quad v = t + 1$

$f' = \frac{(t+1)(2t+6) - (t^2+6t)1}{(t+1)^2}$

$= \frac{2t^2+8t+6-t^2-6t}{(t+1)^2}$

$= \boxed{\frac{t^2+2t+6}{(t+1)^2}}$

8. L(x) = (sin(x) – csc(x) + x)/(tan(x) – 4x^3)

$u = \sin x - \csc x + x \quad v = \tan x - 4x^3$

$L' = \frac{vu' - uv'}{v^2}$

$L' = \boxed{\frac{(\tan x - 4x^3)(\cos x + \csc(x)\cot(x) + 1) - (\sin(x) - \csc(x) + x)(\sec^2(x) - 12x^2)}{(\tan(x) - 4x^3)^2}}$

9. Show that the derivative of sec(x) is sec(x) tan(x).

$f(x) = \sec(x) = \frac{1}{\cos(x)} \qquad u = 1 \ ; \ v = \cos(x)$

$f' = \frac{vu' - uv'}{v^2} = \frac{\cos(x)0 - 1(-\sin(x))}{\cos^2(x)}$

$= \frac{\sin(x)}{(\cos(x))\cos(x)} = \frac{1}{\cos(x)} \, \frac{\sin x}{\cos x}$

$= \boxed{\sec(x)\tan(x)}$

10. What is the derivative of –cos(x) evaluated at π/6 radians?

$f(x) = -\cos(x)$

$f'(x) = \sin(x)$

$f'(\pi/6) = \sin(\frac{\pi}{6})$

$= \boxed{\frac{1}{2}}$

11. If f(x) = 12x^{2}cot(x), find f ’(π radians).

$u = 12x^2 \quad v = \cot(x)$

$f' = uv' + vu'$

$f'(x)$
$= 12x^2(-\csc^2 x) + \cot x \; 24x$

$f'(\pi)$
$= 12\pi^2(-\csc^2(\pi)) + \cot(\pi) \; 24\pi$

undefined

Not possible!

12. Find the equation of the tangent line (at x = 1) to the curve given by f(x) = (x^2 + 4)$(\sqrt{x})^3$.

$f = (x^2+4)x^{3/2}$

$f'(x) = (x^2+4)\frac{3}{2}(x^{1/2}) + x^{3/2}(2x)$

$m = f'(1) = (1^2+4)\frac{3}{2}1^{1/2} + 1^{3/2}(2 \cdot 1)$

$= 5\frac{3}{2} + 2 = 9.5$

$f(1) = (1^2+4)(\sqrt{1})^3 = 5$

$(x_1, y_1) = (1, 5)$

$y - y_1 = m(x - x_1)$

$\boxed{y - 5 = 9.5(x-1)}$

13. Find the equation of the normal line to the curve given by y = 1/x at x = 2.

$f(x) = x^{-1}$

$f'(x) = -1x^{-2} = \frac{-1}{x^2}$

$f'(2) = \frac{-1}{2^2} = \frac{-1}{4} = m_{tan}$

$m_\perp = 4$

$f(2) = \frac{1}{2}$

$(x_1, y_1) = (2, \frac{1}{2})$

$y - y_1 = m(x - x_1)$

$\boxed{y - \frac{1}{2} = 4(x-2)}$

14. What is a linear approximation to the curve $f(x) = \sqrt{x} + x$ at x = 4.1? Is this estimate higher or lower than the true value? Why?

$f(x) = x^{1/2} + x$

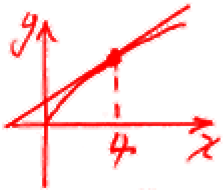
$f'(x) = \frac{1}{2}x^{-1/2} + 1$

$f'(4) = \frac{1}{2\sqrt{4}} + 1$

$m = \frac{1}{4} + 1 = \frac{5}{4}$

$f(4) = \sqrt{4} + 4 = 6$

$(x_1, y_1) = (4, 6)$



use $x=4$

$y - y_1 = m(x - x_1)$

$y - 6 = \frac{5}{4}(x - 4)$

$y = \frac{5}{4}(x - 4) + 6$

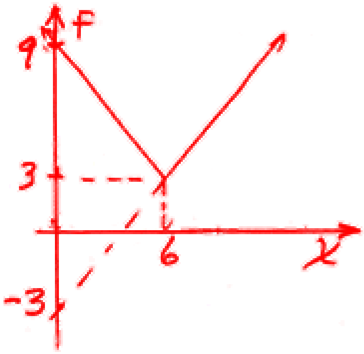
$y(4.1) = \frac{5}{4}(4.1 - 4) + 6$

$= \boxed{6.125}$

Estimate is high. Line is above curve.

---

15. What is the derivative of f(x) = |x – 6| + 3?



$$f(x) = \begin{cases} -x + 9 & x \le 6 \\ x - 3 & x > 6 \end{cases}$$

$$\boxed{f'(x) = \begin{cases} -1 & x < 6 \\ 1 & x > 6 \end{cases}}$$

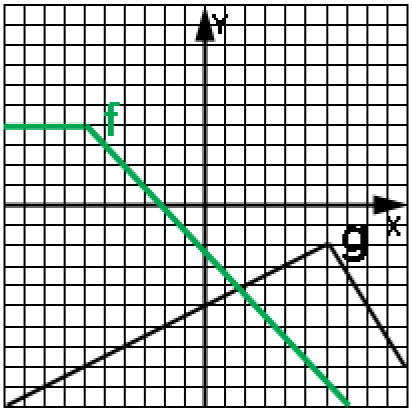
---

16. Find the derivative of the following piecewise function:

$$f(x) = \begin{cases} -x^3 + 2x & \text{if } x < -3 \\ x^2 & \text{if } x \ge -3 \end{cases}$$

$$\boxed{f'(x) = \begin{cases} -3x^2 + 2 & x < -3 \\ 2x & x > -3 \end{cases}}$$

17. Using the functions f(x) and g(x) from the adjacent graph, find p'(–4) where p(x) = f(x) g(x).



$f(-4) = 2$ $f'(-4) = \frac{\text{rise}}{\text{run}} = -1$

$g(-4) = -7$ $g'(-4) = \frac{\text{rise}}{\text{run}} = \frac{1}{2}$

$$p' = f \cdot g' + g \cdot f'$$
$$= 2\left(\tfrac{1}{2}\right) + (-7)(-1)$$
$$= 1 + 7 = \boxed{8}$$