

Blue Pelican
Geometry In-Depth Topics



Teacher Version 1.01

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In-Depth Topic A



Sign Rules

Rules for addition:

If signs are alike: Add the two numbers and take the sign they have.

Examples:

$$3 + (+4) = +7$$

$$5 + 8 = 13$$

$$-9 - 2 = -11$$

$$(-5) - 4 = -9$$

$$-4 + (-6) = -10$$

If signs are different: Subtract and give the answer the sign of the larger number.

Examples:

$$3 + (-7) = -4$$

$$9 - 11 = -2$$

$$14 - 8 = 6$$

$$22 + (-1) = 21$$

Rules for multiplication:

If signs are alike: Multiply and give the answer a positive sign.

Examples:

$$3(4) = 12$$

$$-3(-12) = 36$$

$$(-5)(-3) = 15$$

If signs are different: Multiply and give the answer a negative sign.

Examples:

$$(-3)4 = -12$$

$$5(-2) = -10$$

Rules for division: (essentially the same as for multiplication)

If signs are alike: Divide and give the answer a positive sign.

Examples:

$$12 / (4) = \mathbf{3}$$

$$6 / 2 = \mathbf{3}$$

$$-12 / (-3) = \mathbf{4}$$

$$(-15) / (-3) = \mathbf{5}$$

If signs are different: Divide and give the answer a negative sign.

Examples:

$$(-30) / 5 = \mathbf{-6}$$

$$16 / (-2) = \mathbf{-8}$$

$$-8 / 2 = \mathbf{-4}$$

Assignment: Perform the indicated operations, taking special care with the signs.

1. $5(-3) = -15$
2. $8(5) = 40$
3. $-9 / (-3) = 3$
4. $-2(-6) = 12$
5. $22(-1) = -22$
6. $-12(-2) = 24$
7. $3 + (-8) = -5$
8. $(-50) / 10 = -5$
9. $2 + (+19) = 21$
10. $16(2) = 32$
11. $23 + (-2) = 21$
12. $-8 / 4 = -2$
13. $15 - 6 = 9$
14. $16 / (-2) = -8$
15. $36 / 4 = 9$
16. $(-3)(-8) = 24$
17. $5(-4) = -20$
18. $-3(-22) = 66$
19. $9 - 12 = -3$
20. $5 + (8) = 13$
21. $-6 + (-7) = -13$
22. $8 + (-11) = -3$
23. $(-2) - 4 = -6$
22. $-19(-2) = 38$

In-Depth Topic B



(Derivation of the Quadratic Formula)

Begin with the **general form of the quadratic equation**, $ax^2 + bx + c = 0$ and show that the two solutions are given by the **quadratic formula**:

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

$$\begin{aligned}
 ax^2 + bx + c &= 0 \\
 x^2 + \frac{b}{a}x + \frac{c}{a} &= 0 \quad \text{solve by "completing the square"} \\
 x^2 + \frac{b}{a}x &= -\frac{c}{a} \\
 x^2 + \frac{b}{a}x + \left(\frac{1}{2} \frac{b}{a}\right)^2 &= -\frac{c}{a} + \frac{b^2}{4a^2} \\
 \left(x + \frac{b}{2a}\right)^2 &= -\frac{c}{a} \frac{4a}{4a} + \frac{b^2}{4a^2} \\
 \left(x + \frac{b}{2a}\right)^2 &= \frac{b^2 - 4ac}{4a^2} \\
 \sqrt{\left(x + \frac{b}{2a}\right)^2} &= \pm \sqrt{\frac{b^2 - 4ac}{4a^2}} \\
 x + \frac{b}{2a} &= \pm \frac{\sqrt{b^2 - 4ac}}{2a} \\
 x &= -\frac{b}{2a} \pm \frac{\sqrt{b^2 - 4ac}}{2a} \\
 \boxed{x} &= \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}
 \end{aligned}$$

In-Depth Topic C



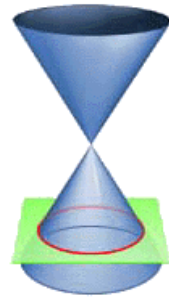
Conic Section Applications and Equation Derivations

Applications of the four conic sections are beautifully illustrated at:

<http://britton.disted.camosun.bc.ca/jbconics.htm>

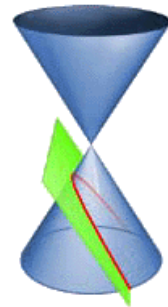
Applications of the **circle**:

No applications are given for this obvious category since they abound in everyday life.



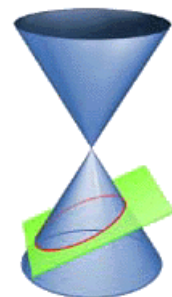
Applications of the **parabola**:

- Some comets pass around the sun in a parabolic path.
- Projectiles follow a parabolic path (a tossed ball).
- The mirrors in flashlights and reflecting telescopes are parabolic in shape.
- The cables of a suspension bridge hang in the shape of a parabola



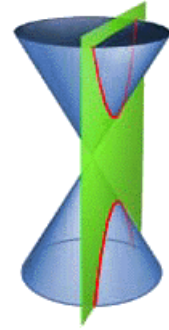
Applications of the **ellipse**:

- Planets (and some comets) orbit about the sun in an elliptical path with the sun at one of the foci.
- Light passing through one focal point will reflect off the ellipse and pass through the other focal point. The same thing happens with sound as evidenced in “whispering galleries.”
- A right circular cylinder sliced at an angle produces an ellipse
- Elliptical gears provide variable rotational speed about an axis passing through a focal point.



Applications of the **hyperbola**:

- Some comets pass around the sun in a hyperbolic path.
- Charged particles fired at heavy nuclei of atoms are deflected into a hyperbolic path.
- Hyperbolic tracking (a navigational technique called Loran) uses intersecting hyperbolas to locate the positions of airplanes and ships.
- The shock wave of a sonic boom intersects the ground in the shape of a hyperbola.
- Hyperbolic as well as parabolic mirrors and lenses are used in telescope systems.



See www.youtube.com/watch?v=XDLyiEWcj_Y for a humorous video about the applications of conic sections.

Equation of a circle (centered at the origin):

A circle is defined as the locus of points that are a fixed distance (called the radius r) from a fixed point called the center of the circle:

The equation is easily obtained by applying the distance formula between two points:

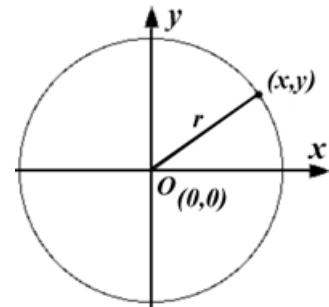
$$d = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$$

Let point #1 be the center at $(0, 0)$ and point #2 be (x, y) (some arbitrary point on the circle) to get:

$$d = \sqrt{(x - 0)^2 + (y - 0)^2}$$

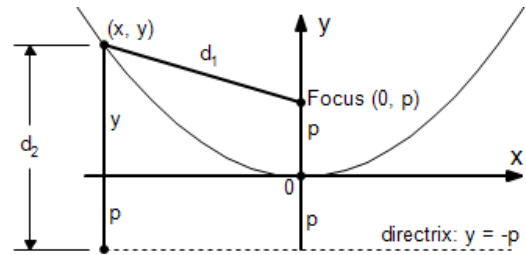
Square both sides, simplify, and let $d = r$ (the radius of the circle) to get:

$$x^2 + y^2 = r^2$$



Equation of a parabola (with vertex at the origin):

A parabola is defined as the locus of points (x, y) that are equidistant from a fixed point called the focus and a fixed line called the directrix:



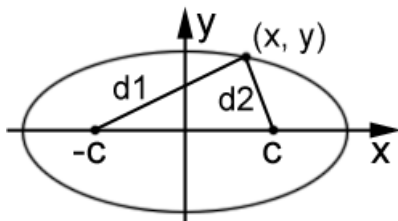
$$d_1 = d_2$$

Apply the distance formula to d_1 and d_2 so that

$$d_1 = \sqrt{(x - 0)^2 + (y - p)^2} \quad \text{and} \quad d_2 = y - (-p)$$

Substituting these into $d_1 = d_2$ and simplifying yields:

$$y = \frac{1}{4p} x^2$$

Equation of an ellipse (centered at the origin):

An ellipse is defined as the locus of points such that the **sum** of the distances from some arbitrary point (x, y) on the ellipse to two fixed points (focal points at $(-c, 0)$ and $(c, 0)$) is a constant:

$$d_1 + d_2 = 2a, \quad \text{where } a \text{ is a constant } > 0.$$

Apply the distance formula to d_1 and d_2 so that

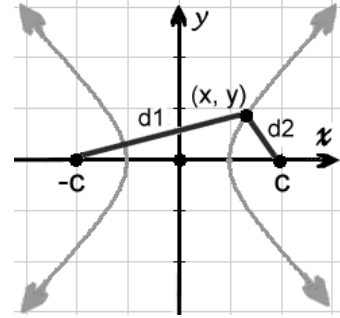
$$d_1 = \sqrt{(x - (-c))^2 + (y - 0)^2} \quad \text{and} \quad d_2 = \sqrt{(x - c)^2 + (y - 0)^2}$$

Substituting these into the top equation, simplifying, and making the substitution $\mathbf{b^2 = a^2 - c^2}$ yields:

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$$

Equation of a hyperbola (centered at the origin):

A hyperbola is defined as the locus of points such that the **difference** of the distances from some arbitrary point (x, y) on the hyperbola to two fixed points (focal points at $(-c, 0)$ and $(c, 0)$) is a constant:



$$d_1 - d_2 = 2a, \text{ where } a \text{ is a constant } > 0.$$

Apply the distance formula to d_1 and d_2 so that

$$d_1 = \sqrt{(x - (-c))^2 + (y - 0)^2} \quad \text{and} \quad d_2 = \sqrt{(x - c)^2 + (y - 0)^2}$$

Substituting these into the top equation, simplifying, and making the substitution $\mathbf{b^2 = c^2 - a^2}$ yields:

$$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$$

The equations for the four conic sections above assume centering at the origin. If a conic section is to be **translated** so as to be centered at (h, k) , substitute $(x - h)$ for x and $(y - k)$ for y in each equation to get:

Circle: $(x - h)^2 + (y - k)^2 = r^2$

Parabola: $y - k = \frac{1}{4p} (x - h)^2$

Ellipse: $\frac{(x-h)^2}{a^2} + \frac{(y-k)^2}{b^2} = 1$

Hyperbola: $\frac{(x-h)^2}{a^2} - \frac{(y-k)^2}{b^2} = 1$

The translated equations of conic section just presented represent just one of several possible **transformations**. These particular transformations preserve relative distances and angles. Some other transformations do not. For example, some transformations distort the original shape by horizontally or vertically stretching it.

It should be noted that the translated conic section equations developed here are in an ideal form that help in identifying the various properties that facilitate creating a graph. For example, consider this equation of a circle:

$$(x - 3)^2 + (y + 5)^2 = 16$$

At a glance, it is easily seen that the center is at (3, -5) and the radius is 4. What about a circle whose equation is given by an equation like this?

$$3x^2 + 3y^2 - 6x + 24y + 24 = 0$$

How would we identify the center and radius? The following example demonstrates the technique of “completing the square” to accomplish this. The technique can also be applied to the other conic sections.

Example: Find the center and radius of the circle represented by this equation and then graph the circle: $3x^2 + 3y^2 - 6x + 24y + 24 = 0$

$$\frac{3x^2}{3} + \frac{3y^2}{3} - \frac{6x}{3} + \frac{24y}{3} + \frac{24}{3} = \frac{0}{3}$$

$$x^2 - 2x + y^2 + 8y = 0 - 8$$

$$x^2 - 2x + 1 + y^2 + 8y + 16 = -8 + 1 + 16$$

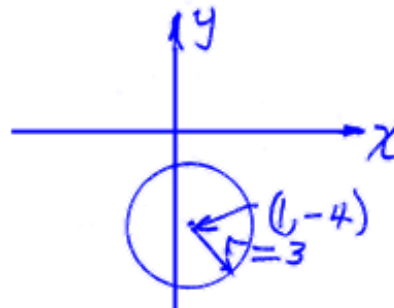
$$\left(\frac{-2}{2}\right)^2 = 1$$

$$\left(\frac{8}{2}\right)^2 = 16$$

$$(x-1)^2 + (y+4)^2 = 9 \text{ circle}$$

$(h, k) = (1, -4)$ center

$r^2 = 9$
 $r = 3$ radius



In-Depth Topic D



Euclidean/non-Euclidean geometry

In this course, we have studied the geometry of flat surfaces as well as simple three-dimensional space. This kind of geometry is called traditional, or Euclidean, geometry. This geometry assumes that the following is true.

Parallel Postulate: Given a line and a point not on the line, there exists one and only one line that goes through the point and is parallel to the original line.

However, there are different kinds of geometry in which the Parallel Postulate is false. These geometries involve working on curved surfaces. We call these geometries non-Euclidean geometries. Some of these geometries modify the Parallel Postulate. Some examples of non-Euclidean geometries are illustrated below.

Spherical Geometry

Spherical geometry is the geometry on the surface of a sphere. Its modified Parallel Postulate is below.

Parallel Postulate (Spherical Geometry): Given a “line” (actually circles) and a point not on the line, no lines exist that go through the point and are parallel to the original line.

An interesting fact about spherical geometry is that the angles of a triangle always add up to be greater than 180 degrees. In addition, the shortest distance between two points in spherical geometry is not necessarily unique.

Hyperbolic Geometry

Hyperbolic geometry is the geometry on the surface of a hyperbolic paraboloid (a saddle-shaped “plane.”) Its modified Parallel Postulate is below.

Parallel Postulate (Hyperbolic Geometry): Given a line and a point not on the line, there exist at least two lines that go through the point and are parallel to the original line.

An interesting fact about hyperbolic geometry is that the angles of a triangle always add up to be less than 180 degrees.

Taxicab Geometry

Taxicab geometry is a modified version of Euclidean geometry in which distances between objects in space are calculated differently. Recall that in Euclidean geometry, the distance between two points is measured using the distance formula:

$$d = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$$

In Taxicab geometry, the distance between two points is the sum of the absolute values of the differences of their coordinates.

$$d = |x_2 - x_1| + |y_2 - y_1|$$

This has many interesting effects; for example, a circle drawn in Taxicab geometry looks like a square.

In-Depth Topic E



Constructions

Copying a line segment... see lesson 3-04

Copying an angle... see lesson 3-04

Perpendicular bisector of a line segment... see lesson 3-04

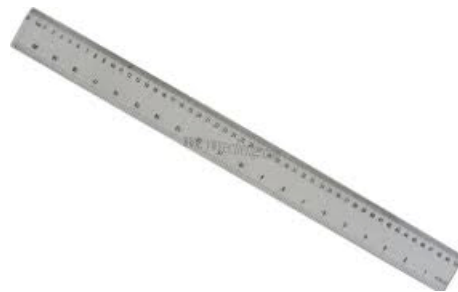
Bisecting an angle... see lesson 3-04

Circumscribing a circle about a triangle... see lesson 15-04

Inscribing a circle inside a triangle and regular polygons... see lesson 16-06



The constructions mentioned above and those that follow in this document are all done with just two drawing tools, a compass and a straight edge (and a sharp pencil, of course).



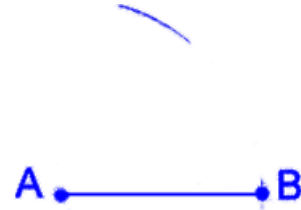
Please note that the following constructions are all fully demonstrated in an associated video.

Constructing an **equilateral triangle**:

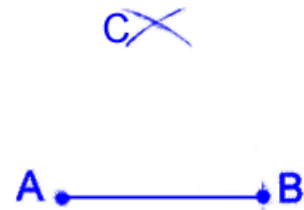
Begin with line segment \overline{AB} and set the width of the compass to AB .



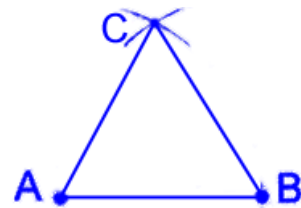
With the point of the compass at point A, strike an arc above the line segment.



Repeat with the point of the compass at point B. Call the intersection of the two arcs point C.

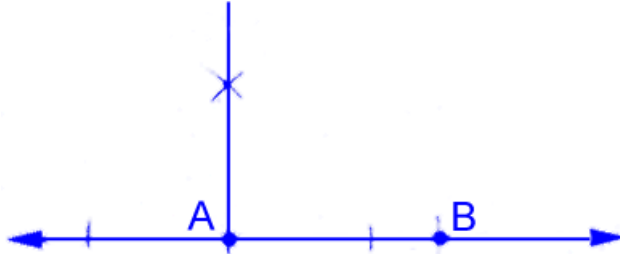


Complete the triangle by drawing line segments \overline{AC} and \overline{BC} .

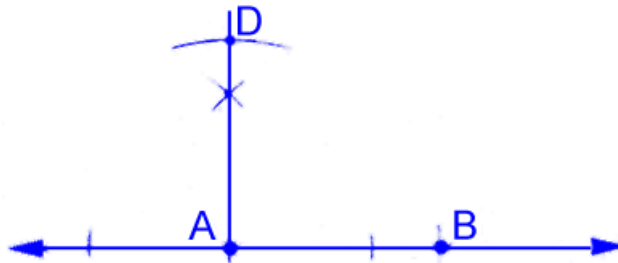


Constructing a **square**:

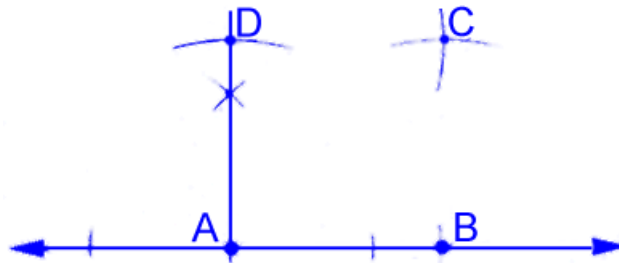
Begin with line \overleftrightarrow{AB} . (From it, we will construct a square having sides of length AB. At point A, construct a perpendicular as was done in Lesson 3-4.



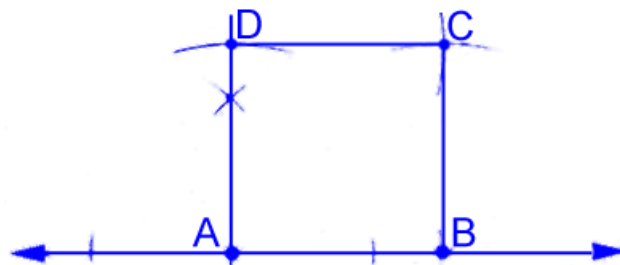
Set the span on the compass to AB and then with the point at A, strike an arc on the perpendicular. Call the point of intersection of the arc and the perpendicular point D.



With the same setting on the compass and with the point at D, strike an arc to the right of D. Then place the compass at point B and strike an arc directly above B, intersecting the previous one. Call their point of intersection C.



Draw line segments \overline{DC} and \overline{CB} , to complete the square.



Inscribing a **hexagon** inside a circle.

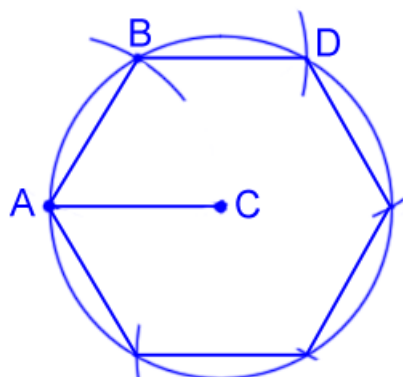
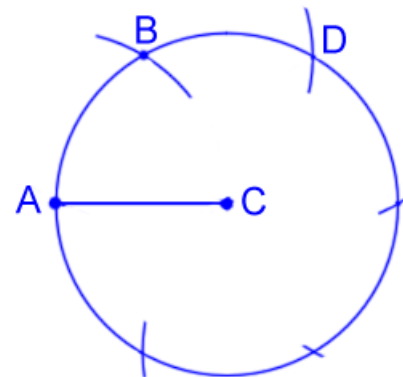
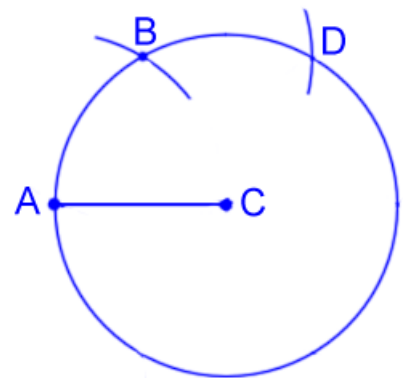
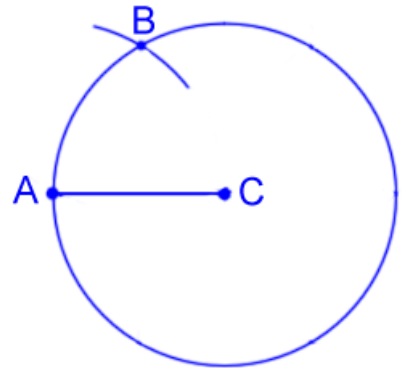
This construction is based on the premise that the central angles of a hexagon are 60° and that the angles of an equilateral triangle are also 60° .

Beginning with a circle of radius AC , construct an equilateral triangle with its base on \overline{AC} . Set the compass span to AC and with the point at A , strike an arc on the circle at B . Triangle ABC is equilateral; therefore, $m\angle ACB = 60^\circ$.

\overline{AB} is one side of the hexagon. With the point of the compass at A and still with a span of AC , strike an arc on the circle at D to establish the next vertex of the hexagon.

Continue in this way, placing the point of the compass at each new vertex and striking an arc on the circle to mark the position of the next vertex.

Connect the marked vertices with line segments to complete the hexagon.



In-Depth Topic F



Exterior Angle Sum Theorem

Prior Knowledge:

Triangle Sum Theorem: *The interior angles of a triangle sum to 180° .*

Supplementary Angle Theorem: *The interior angle of a polygon at any vertex is supplementary to its exterior angle.*

Concept:

Exterior Angle Sum Theorem: *The sum of the exterior angles of a convex polygon with N angles is equal to 360° .*

Method:

Look at the following figure showing the exterior angles of a convex, regular polygon. It can be seen that the exterior angles must sum to 360° . This is shown without proof by using a sequence of dilations (Fig. 1), and then with an example (Fig. 2).

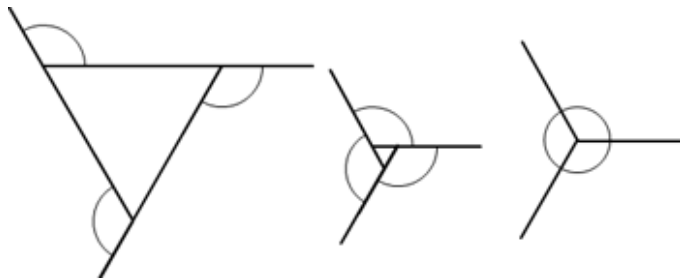


Fig. 1

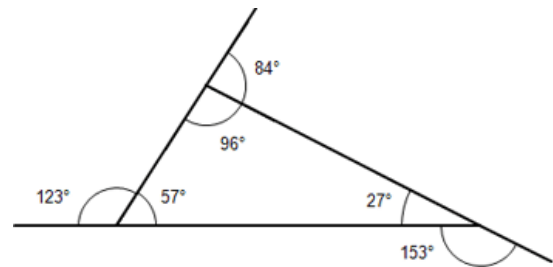


Fig. 2

Consider the following table of exterior angles of the first few regular, convex polygons:

Name	No. of Angles (N)	Exterior Angle (E) ($360^\circ/N$)	Exterior Angle Sum $N*(E)$
Triangle	3	120°	360°
Square	4	90°	360°
Pentagon	5	72°	360°
Hexagon	6	60°	360°

Corollary:

Working backward from the table, one is able to derive a rule for finding the measure of one exterior angle of a regular convex polygon with N angles: $E = 360^\circ/N$.

In-Depth Topic G



Interior Angle Sum Theorem

Prior Knowledge:

Exterior Angle Sum Theorem: *The sum of the exterior angles of a polygon with N angles is always equal to 360° .*

Supplementary Angle Theorem: *The interior angle of a polygon at any vertex is supplementary to its exterior angle.*

Corollary to the Exterior Angle Sum Theorem:

Interior Angle Sum Theorem: The sum of the interior angles of a polygon with N angles is $(N-2)180^\circ$.

Method:

Look at the following figure (Fig. 1) showing the interior angles of a convex, regular polygon. It will be seen that the interior angles must be the sum of the number of triangles that can be drawn from one vertex within the polygon, and that this number is always two less than the number of angles. This is shown without proof (Fig. 2).

Shown below is a table the measures of the interior and exterior angles of some regular convex polygons. The exterior angle sums have been omitted, as they are always 360° .

Name	No. of Angles (N)	Exterior Angle (E) ($360/N$)	Interior Angle (I) ($180-E$)	Interior Angle Sum (S) $N*(I)$
Triangle	3	120	60	180
Square	4	90	90	360
Pentagon	5	72	108	540
Hexagon	6	60	120	720

Look at the Interior Angle Sum column: It is just like adding an extra triangle to the previous polygon each time. This concept is illustrated in the drawing below. Note how 180° is added to the interior angle sum each time.

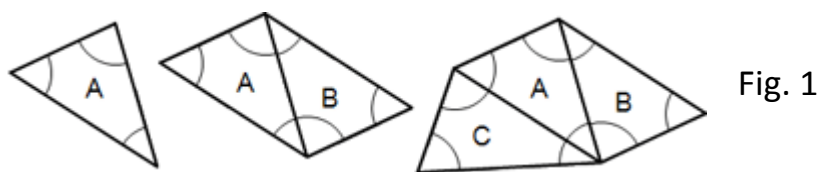


Fig. 1

Notice that adding a triangle is adding one additional vertex. This gives us the following relation:

No. Angles	3	4	5	6	7	8
No. Triangles	1	2	3	4	5	6

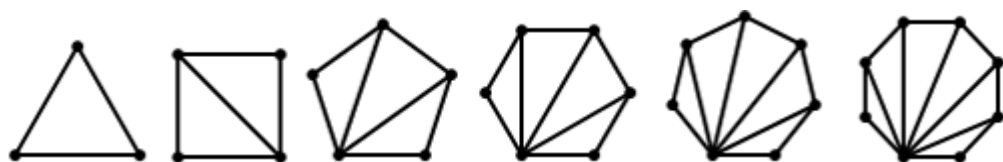


Fig. 2

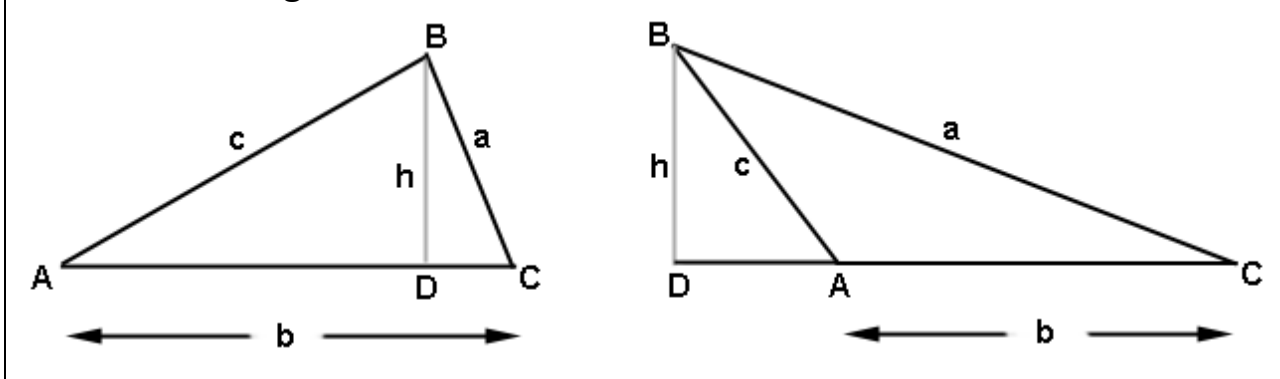
Working backward from the tables, one is able to derive a rule for finding the measure of the sum of the interior angles of a convex polygon with N angles: $S = (N-2)180^\circ$. Also, it is possible to find a relation between the number of angles (N) and the measure of one interior angle of a convex, regular polygon: $I = 180^\circ - 360^\circ/N$

In-Depth Topic H



(Derivation of the Sine Law)

Consider the following two triangles in which a perpendicular has been dropped from vertex B to side AC. The reason we consider two triangles is so that we cover both cases: when the perpendicular falls **inside** the triangle and when it falls **outside**.



From either figure above, solve for h from triangle BCD in terms of a and C :

$$\sin C = \frac{h}{a} ; h = a \sin C$$

Using either triangle ABD from the left figure or triangle ABD from the right figure, solve for h in terms of c and A (Notice in the right picture, angle DAB and angle CAB have the same sine.):

$$\sin A = \frac{h}{c} ; h = c \sin A$$

Set the two equations for h equal to each other:

$$a \sin C = c \sin A$$

$$\div ac \quad \rightarrow \quad \boxed{\frac{\sin C}{c} = \frac{\sin A}{a}} \quad \text{Sine Law}$$

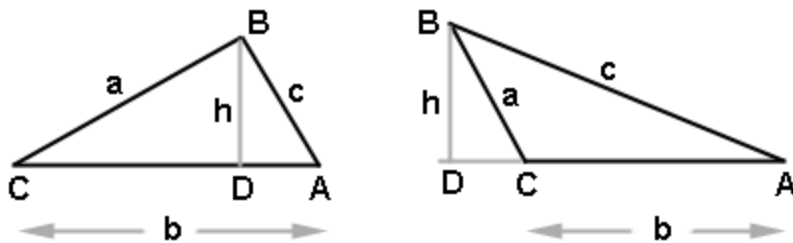
Similarly, it can be proved that $\frac{\sin A}{a} = \frac{\sin B}{b}$ and $\frac{\sin C}{c} = \frac{\sin B}{b}$

In-Depth Topic I



(Derivation of the Cosine Law)

Consider triangle ABC in which a perpendicular is dropped from B to the line described by opposite side. The left drawing below is when this perpendicular falls inside the triangle and the right picture shows it falling outside the triangle. Both scenarios will be considered in this derivation.



Applying the Pythagorean theorem to either right triangle, we get:

$$(1) \quad c^2 = h^2 + (DA)^2$$

Solve for h from either right triangle and get:

$$(2) \quad h = a \sin C$$

Solve for DA in terms of a , b , and C

$$(3) \quad DA = b - \underbrace{a \cos C}_{CD}$$

Notice $\cos C$ is ~~neg~~ for the right picture which makes this term positive

Now substitute equations (2) and (3) into equation (1) and get:

$$\begin{aligned}c^2 &= (a \sin C)^2 + (b - a \cos C)^2 \\&= a^2 \sin^2 C + b^2 - 2ab \cos C + a^2 \cos^2 C \\&= a^2 (\underbrace{\sin^2 C + \cos^2 C}_1) + b^2 - 2ab \cos C\end{aligned}$$

$$c^2 = a^2 + b^2 - 2ab \cos C$$

Cosine law

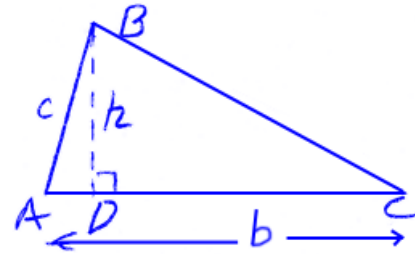
In-Depth Topic J



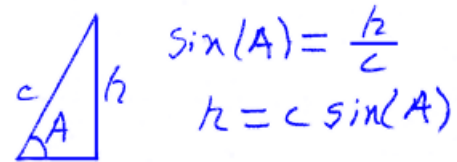
Derivation of a Triangle Area Formula

The area of a triangle is given by the multiplying one-half, the product of two sides, and the sine of their included angle.

To derive this formula, begin with an acute triangle ABC and with b as the base. Draw a perpendicular from vertex B to the base b and label this line segment as h , the height of the triangle. Label the point of intersection of the perpendicular with the base as point D.



Consider right triangle ABD. Find h in terms of AB and $\angle A$.

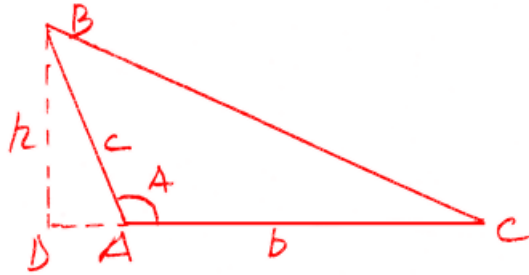


Now find the area of the triangle using the formula $(1/2)(\text{base})(\text{height})$.

$$\begin{aligned} \text{Area} &= \frac{1}{2} (\text{Base})(\text{height}) \\ &= \boxed{\frac{1}{2} b c \sin(A)} \end{aligned}$$

Assignment:

Duplicate the derivation just done, this time using an obtuse triangle.



$$\sin(\angle A) = \sin(\angle DAB)$$

$$\sin(A) = \frac{h}{c}$$

$$h = c \sin(A)$$

$$\text{Area} = \frac{1}{2} (\text{base})(\text{height})$$

$$= \frac{1}{2} (b) c \sin(A)$$

In-Depth Topic K



Analytic Geometry and the use of Equations in Geometry

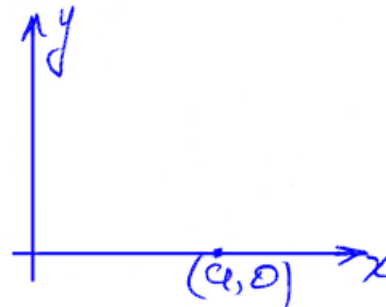
Analytic geometry (also known as coordinate geometry) makes extensive use of coordinates and equations. Three examples are given here that distinguish analytic geometry from “regular” geometry:

- To prove that two lines are parallel, find the slopes of the lines using $m = \frac{y_2 - y_1}{x_2 - x_1}$ and show they are equal.
- To prove that two lines are perpendicular, find the slopes of the lines and show that their product is -1.
- To show that two line segments are congruent, find the lengths of each using the distance formula $d = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$ and show they are equal.

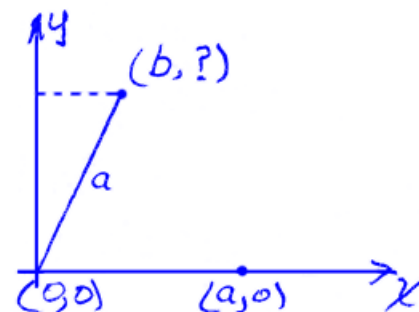
Two theorem proofs are offered here illustrating the use of analytic geometry:

Theorem: The diagonals of a rhombus are perpendicular to each other.


First, draw a rhombus on a coordinate system in which the coordinates are as general as possible. Begin with an initial side being the line segment between $(0, 0)$ and $(a, 0)$. Thus, we are committed to a rhombus with sides of length a .



Now draw a second side of the rhombus (of length a) in which the side is “leaned” out by a general amount b .



Next, find the y value of this point by an application of the Pythagorean theorem.

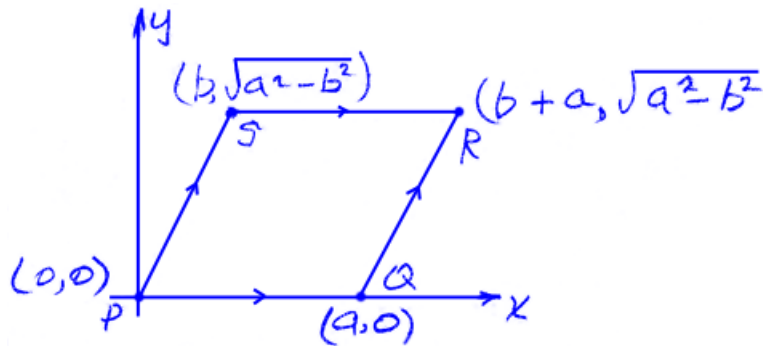


$$y^2 + b^2 = a^2$$

$$y^2 = a^2 - b^2$$

$$y = \sqrt{a^2 - b^2}$$

The final point of the rhombus is now found by moving a distance a to the right of the last point found. Label the four vertices of the rhombus as P, Q, R and S.



Find the slope of \overline{PR} (call it m_{PR}):

$$m_{PR} = \frac{y_R - y_P}{x_R - x_P} = \frac{\sqrt{a^2 - b^2} - 0}{b + a - 0} = \frac{\sqrt{a^2 - b^2}}{b + a}$$

Find the slope of \overline{QS} (call it m_{QS}):

$$m_{QS} = \frac{y_S - y_Q}{x_S - x_Q} = \frac{\sqrt{a^2 - b^2} - 0}{b - a} = \frac{\sqrt{a^2 - b^2}}{b - a}$$

The proof is completed by showing that $m_{PR}(m_{QS}) = -1$.

$$m_{PR} m_{QS} = \frac{\sqrt{a^2 - b^2}}{b + a} \cdot \frac{\sqrt{a^2 - b^2}}{b - a} = \frac{a^2 - b^2}{b^2 - a^2}$$

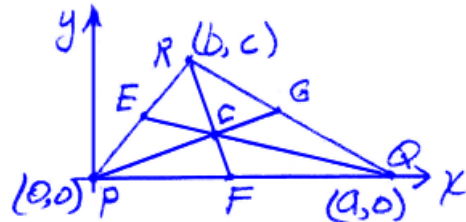
$$= \frac{a^2 - b^2}{-1(a^2 - b^2)} = \boxed{-1}$$

Theorem: The medians of a triangle all meet in a common point and that point divides each median into two line segments having the ratio 1:2.

Recall that a **median** of a triangle is the line segment drawn from a vertex to the midpoint of the opposite side.

The common point of intersection of all three medians is called the **centroid** of the triangle.

Begin by drawing a general (non-special) triangle as shown here (label it PQR). Assign general coordinates to the vertices. Also mark and label the midpoints as shown:

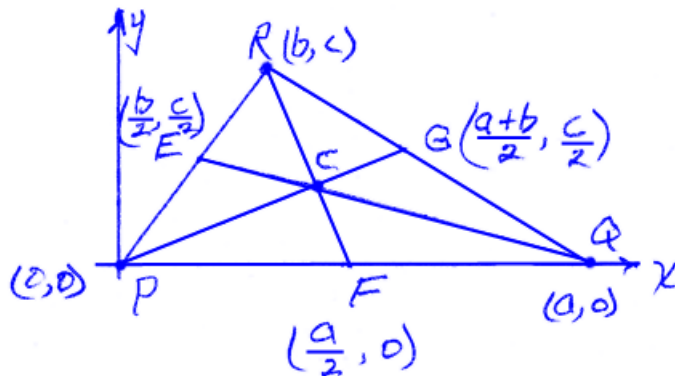


Determine the coordinates of the midpoints E, F and G using the generalized coordinates of the line segments upon which they lie. Then label them on the drawing.

$$E_x = \frac{b+0}{2} = \frac{b}{2} ; E_y = \frac{c+0}{2} = \frac{c}{2} ; E \rightarrow \left(\frac{b}{2}, \frac{c}{2} \right)$$

$$F_x = \frac{a+0}{2} = \frac{a}{2} ; F_y = 0 ; F \rightarrow \left(\frac{a}{2}, 0 \right)$$

$$G_x = \frac{a+b}{2} ; G_y = \frac{c+0}{2} = \frac{c}{2} ; G \rightarrow \left(\frac{a+b}{2}, \frac{c}{2} \right)$$



The remainder of the proof is left to the student in the assignment that follows.

Assignment: Continue the proof started on the preceding page by performing these steps:

- Using the coordinates of points R and F, find the equation of the line passing through R and F.

$$m = \frac{c-0}{b-\frac{a}{2}} \cdot \frac{2}{2} = \frac{2c}{2b-a} \leftarrow \text{rise/run}$$

$$y-0 = \frac{2c}{2b-a} \left(x - \frac{a}{2}\right) \leftarrow \text{pt slope Form}$$

$$\boxed{y = \frac{2c}{2b-a} \left(x - \frac{a}{2}\right)}$$

- Using the coordinates of points P and G, find the equation of the line passing through P and G.

$$m = \frac{\frac{c}{2}-0}{\frac{a}{2}+b} = \frac{c}{2} \cdot \frac{2}{a+b} = \frac{c}{a+b} \leftarrow \text{rise/run}$$

$$y-0 = \frac{c}{a+b}(x-0) \leftarrow \text{pt slope Form}$$

$$\boxed{y = \frac{c}{a+b}x}$$

- Using the coordinates of points Q and E, find the equation of the line passing through Q and E.

$$m = \frac{\frac{c}{2}-0}{\frac{b}{2}-a} \cdot \frac{2}{2} = \frac{c}{b-2a} \leftarrow \text{rise/run}$$

$$y-0 = \frac{c}{b-2a}(x-a) \leftarrow \text{pt. slope form}$$

$$\boxed{y = \frac{c}{b-2a}(x-a)}$$

4. Solve the equations for \overrightarrow{RF} and \overrightarrow{PG} to find their intersection point.

$$y = \frac{c}{a+b}x ; y = \frac{2c}{2b-a}\left(x - \frac{a}{2}\right)$$

$$\frac{c}{a+b}x = \frac{2c}{2b-a}\left(x - \frac{a}{2}\right)$$

$$\frac{1}{a+b}x = \frac{2}{2b-a}x - \frac{a}{x(2b-a)}$$

$$\frac{1}{a+b}x - \frac{2}{2b-a}x = -\frac{a}{(2b-a)} \quad \text{mult both sides by}$$

$$(2b-a)x - 2(a+b)x = -a(a+b) \quad (a+b)(2b-a)$$

$$(\cancel{2b-a} - 2a - \cancel{2b})x = -a(a+b)$$

$$-3ax = -a(a+b)$$

$$x = \frac{a+b}{3}$$

$$C = \left(\frac{a+b}{3}, \frac{c}{3}\right)$$

$$y = \frac{c}{a+b}x$$

$$y = \frac{c}{a+b} \frac{a+b}{3}$$

$$y = \frac{c}{3}$$

5. Solve the equations for \overrightarrow{EQ} and \overrightarrow{PG} to find their intersection point.

$$y = \frac{c}{b-2a}(x-a) ; y = \frac{c}{a+b}x$$

$$\frac{c}{b-2a}(x-a) = \frac{c}{a+b}x$$

$$\frac{1}{b-2a}x - \frac{a}{b-2a} = \frac{1}{a+b}x$$

$$\frac{1}{b-2a}x - \frac{1}{a+b}x = \frac{a}{b-2a}$$

$$x\left(\frac{1}{b-2a} - \frac{1}{a+b}\right) = \frac{a}{b-2a} \quad \text{mult by } (b-2a)(a+b)$$

$$x(a+b - \cancel{b} + 2a) = a(a+b)$$

$$3ax = a(a+b)$$

$$x = \frac{a+b}{3}$$

$$C = \left(\frac{a+b}{3}, \frac{c}{3}\right)$$

$$y = \frac{c}{a+b}x$$

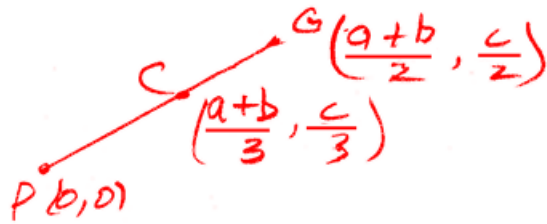
$$y = \frac{c}{a+b} \frac{a+b}{3}$$

$$y = c/3$$

Similarly, solving the remaining pair of equations (\overrightarrow{RF} and \overrightarrow{QE}) would also result in finding $((a+b)/3, c/3)$ as their intersection point.

Now we turn our attention to proving the 2:1 ratio part of the theorem.

6. Find PC and CG and show that PC/PG = 2:1.



$$PC = \sqrt{\left(\frac{a+b}{3} - 0\right)^2 + \left(\frac{c}{3} - 0\right)^2}$$

$$= \sqrt{\frac{(a+b)^2}{9} + \frac{c^2}{9}} = \frac{\sqrt{(a+b)^2 + c^2}}{3}$$

$$CG = \sqrt{\left(\frac{a+b}{2} - \frac{a+b}{3}\right)^2 + \left(\frac{c}{2} - \frac{c}{3}\right)^2}$$

$$= \sqrt{\left(\frac{3a+3b-2a-2b}{6}\right)^2 + \left(\frac{3c-2c}{6}\right)^2}$$

$$= \sqrt{\left(\frac{a+b}{6}\right)^2 + \left(\frac{c}{6}\right)^2}$$

$$= \frac{\sqrt{(a+b)^2 + c^2}}{6}$$

$$\frac{PC}{CG} = \frac{\frac{\sqrt{(a+b)^2 + c^2}}{3}}{\frac{\sqrt{(a+b)^2 + c^2}}{6}} = \frac{2}{1} = \boxed{2:1}$$

Similarly, it can be shown that RC/CF = 2:1 and that QC/CE = 2:1.

In-Depth Topic L



Area & volume density (and associated unit conversions)

In this topic we will look at two types of density:

- Area density – the count of some quantity **per unit area**.
- Volume density – the count of some quantity **per unit volume**.

Before looking at examples of density, first consider the word “per” (notice it’s used in both densities mentioned above).

As a general rule, when the word “per” is used in a mathematical context, the quantity that follows “per” will be placed in the **denominator** of an expression.

For example, “miles per hour” is written as $\frac{\text{miles}}{\text{hour}}$.

“Dollars per dozen” is written as $\frac{\text{dollars}}{\text{dozen}}$.

Example 1: If 87 germs are counted in .2 square centimeters on a slide under a microscope, what is the density of germs per square centimeter? Is this an area or volume density problem?

$$\frac{87 \text{ germs}}{.2 \text{ cm}^2} = \boxed{435 \frac{\text{germs}}{\text{cm}^2}} \quad \text{Area density}$$

Example 2: Convert the answer of example 1 into germs/m².

$$\begin{aligned} \frac{435 \text{ germs}}{1 \text{ cm}^2} \left(\frac{100 \text{ cm}}{1 \text{ m}}\right)^2 &= \frac{435 \text{ germs}}{1 \text{ cm}^2} \frac{100^2 \text{ cm}^2}{1 \text{ m}^2} \\ &= 435 \cdot 100^2 \frac{\text{germs}}{\text{m}^2} = \boxed{4,350,000 \frac{\text{germs}}{\text{m}^2}} \end{aligned}$$

Example 3: At the end of a driving range, 435 golf balls are found in a 5m X 11m area. What is the density of golf balls per square meter?

$$\frac{435 \text{ balls}}{5(11) \text{ m}^2} = \boxed{7.90 \frac{\text{balls}}{\text{m}^2}}$$

Example 4: With the ball density obtained in example 3, how many golf balls could be expected in an area of 100 X 50 m?

$$7.90 \frac{\text{balls}}{\text{m}^2} (100 \cdot 50) \text{ m}^2 = \boxed{39,545.45 \text{ balls}}$$

Example 5: The weight density of a substance is the weight per unit volume. If a cube (1 ft on each side) of gold weighs approximately 1200 lb, what is the density in ounces/in³?

$$\begin{aligned} \frac{1200 \text{ lb}}{1 \text{ ft}^3} \left(\frac{1 \text{ ft}}{12 \text{ in}} \right)^3 &= \frac{1200 \text{ lb} \cancel{\text{ft}^3}}{1 \cancel{\text{ft}^3} 12^3 \text{ in}^3} = \frac{1200 \text{ lb}}{1 \cdot 12^3 \text{ in}^3} \frac{16 \text{ oz}}{1 \text{ lb}} \\ &= \frac{1200(16) \text{ oz}}{12^3 \text{ in}^3} \\ &= \boxed{11.7 \frac{\text{oz}}{\text{in}^3}} \end{aligned}$$

Example 6: People with allergies are generally interested in the daily pollen count. Pollen counts are typically given as grains/m³ with 30 grains/m³ being considered low and above 120 grains/m³ considered high. If a volume of 96 liters of air was found to contain 14 grains of pollen, would this be considered to be a low, medium, or high pollen count? (1 liter = 1000 cm³)

$$\begin{aligned}
 & 14 \frac{\text{grains}}{96 \cancel{\text{L}}} \frac{1 \cancel{\text{K}}}{1000 \text{cm}^3} \left(\frac{100 \text{cm}}{1 \text{m}} \right)^3 \\
 &= \frac{14}{96} \frac{\text{grains}}{\cancel{\text{L}}} \frac{1}{1000 \cancel{\text{cm}^3}} \frac{100^3 \cancel{\text{cm}^3}}{1 \text{m}^3} \\
 &= \boxed{145.8\bar{3} \frac{\text{grains}}{\text{m}^3} \text{ High}}
 \end{aligned}$$

Assignment:

1. In an infestation of locusts, a biologist counted 4 locusts/ft² on the ground. His report requires that he give the density in terms of locusts per square yard. Do the conversion that he needs for his report.

$$\begin{aligned} \frac{4 \text{ locusts}}{\text{ft}^2} \left(\frac{3 \text{ ft}}{1 \text{ yd}}\right)^2 &= \frac{4 \text{ locusts}}{\cancel{\text{ft}^2}} \frac{9 \cancel{\text{ft}^2}}{\text{yd}^2} \\ &= \boxed{36 \frac{\text{locusts}}{\text{yd}^2}} \end{aligned}$$

2. In the manufacture of silicon wafers for the computer-chip industry, a technician counted 3 defects per square mm. What is the corresponding number of defects per square centimeter? At this rate, how many defects will be found on a circular wafer 8 cm in diameter?

$$\begin{aligned} \frac{3 \text{ defects}}{1 \text{ mm}^2} \left(\frac{10 \text{ mm}}{1 \text{ cm}}\right)^2 &= \frac{3 \text{ defects}}{\cancel{1 \text{ mm}^2}} \frac{100 \cancel{\text{mm}^2}}{1 \text{ cm}^2} \\ &= \boxed{300 \frac{\text{defects}}{1 \text{ cm}^2}} \quad \text{area } r = 8 \text{ cm} / 2 = 4 \text{ cm} \\ 300 \frac{\text{defects}}{\text{cm}^2} \pi (4 \text{ cm})^2 &= \frac{300 \text{ defects}}{\cancel{\text{cm}^2}} 16\pi \cancel{\text{cm}^2} \\ &= \boxed{15,079.6447 \text{ defects}} \end{aligned}$$

3. Several years ago, an average of 123 people were found to be living in a square area 100m X 100m in Calcutta, India. What was the corresponding population density in people/km²?

$$\begin{aligned} \frac{123 \text{ people}}{100^2 \text{ m}^2} \left(\frac{1000 \text{ m}}{1 \text{ km}}\right)^2 &= \frac{123 \text{ people}}{100 \cancel{\text{m}^2}} \frac{1000^2 \cancel{\text{m}^2}}{1 \text{ km}^2} \\ &= \frac{123 \cdot 1000^2}{100} \frac{\text{people}}{\text{km}^2} = \boxed{1,230,000 \frac{\text{people}}{\text{km}^2}} \end{aligned}$$

4. Four mosquitos were found in a cubic foot of air. What was the density of mosquitos per cubic yard?

$$\frac{4 \text{ mosq}}{1 \text{ ft}^3} \left(\frac{3 \text{ ft}}{1 \text{ yd}}\right)^3 = \frac{4 \text{ mosq}}{1 \text{ ft}^3} \frac{27 \text{ ft}^3}{\text{yd}^3} = \boxed{108 \frac{\text{mosq}}{\text{yd}^3}}$$

5. What is the mass density of a substance if 3 cm³ of the substance has a mass of 8 grams?

$$\frac{8 \text{ grams}}{3 \text{ cm}^3} = \boxed{2.\bar{6} \text{ g/cm}^3}$$

6. The mass density of aluminum is 2.7gm/ml. If the mass of a particular piece of aluminum is 75 gm, what is its volume?

$$2.7 \frac{\text{gm}}{\text{ml}} \xrightarrow{\text{invert}} \frac{1}{2.7} \frac{\text{ml}}{\text{gm}}$$

$$\frac{1}{2.7} \frac{\text{ml}}{\text{gm}} \frac{75 \text{ gm}}{1} = \boxed{27.7 \text{ ml}}$$

In-Depth Topic M



Deductive and Inductive Reasoning

Deductive reasoning is reasoning based on facts.

- Facts in mathematics are written in the form of definitions, postulates and theorems. Deductive reasoning in mathematics uses these facts to come up with more mathematical facts.
- Deductive reasoning takes **general knowledge** and uses it to discover a **specific** property. When creating a proof in geometry, deductive reasoning is used.
- Deductive reasoning (if correctly done) **always** leads to a correct conclusion provided the facts used were true.

Example 1: Use deductive reasoning to answer the following question.

1. Mr. Brooks drives to work when it is raining; otherwise, he rides his bike.
2. It is raining today.

Based on the true statements above, how did Mr. Brooks get to work?

He drove.

Inductive Reasoning is reasoning based on past events. We use inductive reasoning to formulate mathematical patterns.

- Inductive reasoning takes **specific** events and uses them to create a **general** pattern. When trying to formulate a hypothesis, inductive reasoning is used.
- Since inductive reasoning is based on past events only, it will not always lead to a correct conclusion.

Example 2: Use inductive reasoning to answer the following question.

Give a mathematical rule for the following sequence: {1, 4, 9, 16, 25, 36...}

$$S_n = n^2$$

In-Depth Topic N



Area of a Regular Polygon by Apothem and Perimeter

The area of a regular, convex polygon is equal to one-half the product of its apothem and perimeter.

$$A = \frac{1}{2}(\text{apothem})(\text{perimeter}), \text{ OR, } A = \frac{1}{2}aP$$

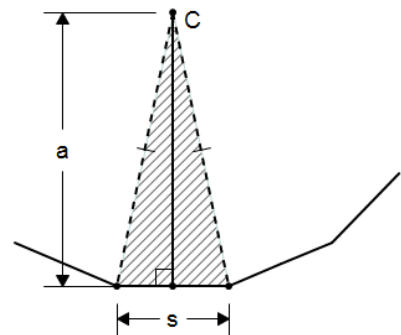
This is an equivalent way of saying that the total area of a regular, convex polygon is equal to the area of one triangular section times the number of sections N :

$$A_{\text{total}} = A_{\text{triangle}}N.$$

Shown here is one of N triangular section in a polygon of N sides. The area of this triangle is

$$\frac{1}{2}h(\text{base}) = \frac{1}{2}a(s)$$

Where a (the apothem) = h and s = side length of the polygon.



To get the total area of the polygon, multiply by N to get:

$$A_{\text{total}} = \frac{1}{2}a(s)N$$

Since $(s)N$ is the perimeter we can finally write

$$A = \frac{1}{2}(\text{apothem})(\text{perimeter}), \text{ or, } A = \frac{1}{2}aP$$

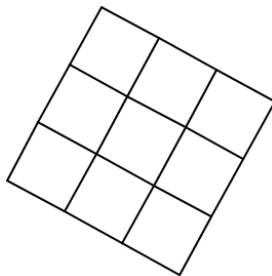
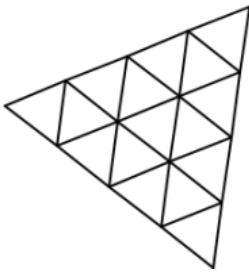
In-Depth Topic O



Tessellations

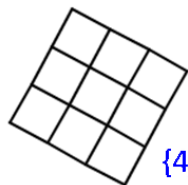
The word **tessellate** comes to English from the Latin word *tessella*, the name for the small square tiles used in mosaics. A **tessellation** is a tiling of a plane with individual polygons, or with groups of polygons called **prototiles**.

- Three regular polygons can tile the Euclidean plane without leaving any gaps or overlapping. These are triangles, squares, and hexagons, which make up the **regular tessellations**.

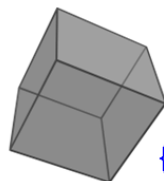


- Notice how at every vertex the angle sum is 360° .
- One way to name tessellations is by identifying the **vertex figure** $\{s,n\}$, the polygons that meet at a vertex, where **s** is the number of sides in the polygon and **n** is the number of polygons.

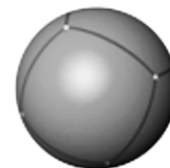
Example 1: Identify these regular tessellations by their vertex figure, and state the geometry of the plane being tiled, either the Euclidean, Hyperbolic, or Spherical plane.



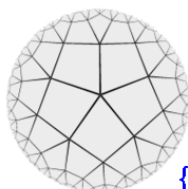
$\{4,4\}$ Euclidean



$\{4,3\}$ Euclidean



$\{4,3\}$ Spherical



$\{4,5\}$ Hyperbolic

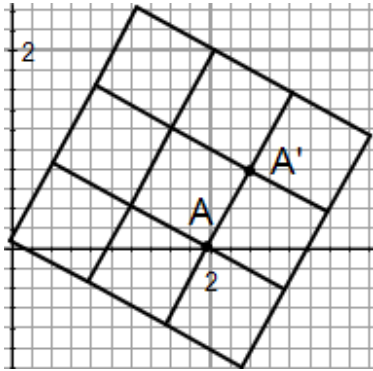


$\{3,6\}$ Euclidean

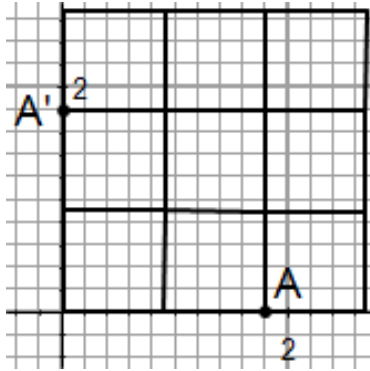


$\{6,3\}$ Euclidean

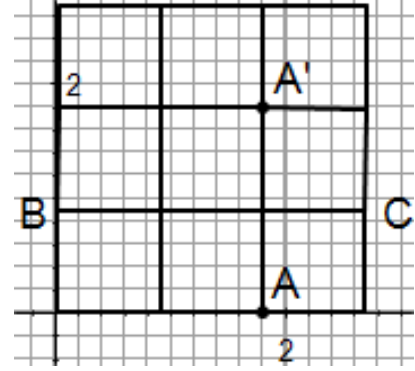
Example 2: Name the isometry that maps vertex A to A'.



$T_{x,y}(0.2,0.4)$



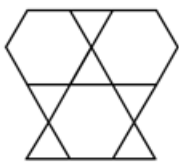
$R_{(0,0)} 90$



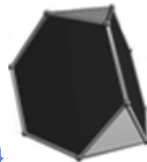
r_{BC}

- Semi-regular tiling uses the convention $\{s_1, s_2, s_3, \dots, s_n\}$ to describe the vertex configuration, where s is the number of sides in each polygon and n are the polygons that meet at the vertex, starting with the lowest number and continuing either CW or CCW.
- Semi-regular tessellations are also vertex-transitive.

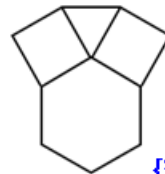
Example 3: Identify these semi-regular tessellations by their vertex configuration:



$\{3,6,3,6\}$



$\{3,6,6\}$



$\{3,4,6,4\}$



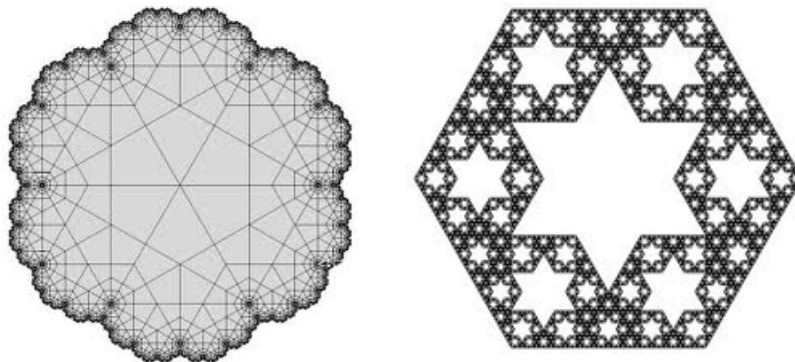
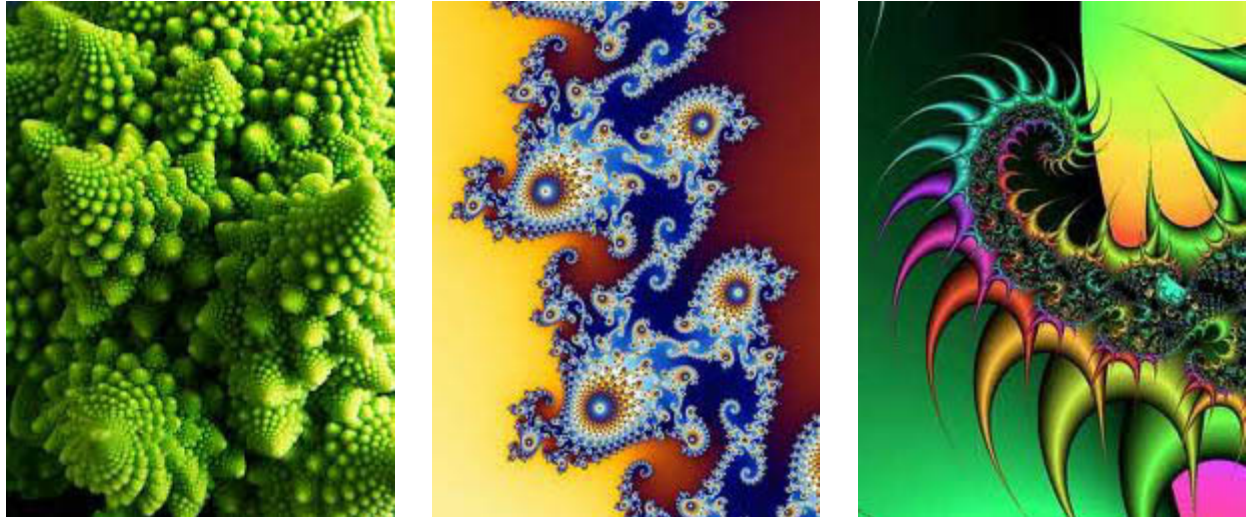
$\{3,4,5,4\}$



In-Depth Topic P

Fractals

Fractals have been described as “**useful beauty**” as attested by these photos of computer generated pictures of fractals:



Background

Fractals is a relatively new branch of mathematics that had its simple beginnings in the 17th century and was primarily and originally based on simple recursion. The subject moved with increasing rigor into the 19th century; however, the word “fractal” was not coined until 1975 by Benoit Mandelbrot, the most famous name associated with fractals. It was not until the present century that interest in fractals really grew into the major field it is today.

What is a fractal?

There is no universal agreement on exactly how to define fractals; however, the general consensus is that they are **infinitely self-similar** and **iterated**. This means that no matter how deeply we zoom on a fractal image, it **always looks the same** and the same construct is continually repeated.

While classical Euclidean geometry assigns a dimension of one to a line, two dimensions to an object in a plane, and three dimensions to objects having length, width, and depth, fractals can have fractional dimensions such as .6309, 1.2619, etc. The formula that determines the dimension of a fractal is given by the following where P is the number of “pieces” and M is the “magnification”:

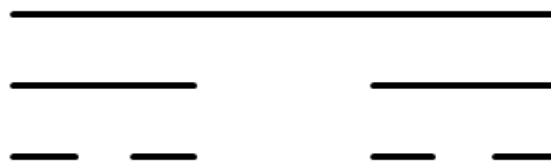
$$\ln(P)/\ln(M)$$

Initiators and Generators

Very simply, the initiator is a starting shape and the generator is the set of rules (an algorithm) that governs the production of the next iteration of the fractal.

The Cantor Set:

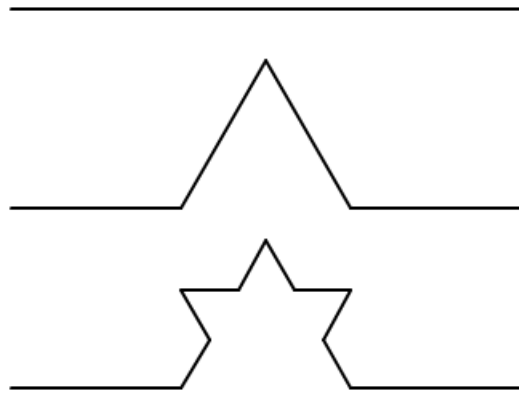
The initiator is a line segment and the generator is the algorithm: “With each iteration, the middle third of the each line segment of the previous set is removed.” Three iterations are shown below:



To calculate the dimension of the Cantor set, notice the magnification M is 3 (the minimum magnification required to see that the fractal is self-similar) and P is 2 (each segment decomposes into 2 smaller segments). The dimension of this fractal is $\ln(P)/\ln(M) = \ln(2)/\ln(3) = .6309$.

The Koch Curve:

The initiator is a line segment of unit length divided into three equal line segments. The generator is the algorithm, “With each iteration, the middle section is replaced with an equilateral triangle that has its base removed.”



To calculate the dimension of the Koch Curve, notice the magnification M is 3 (the minimum magnification required to see that the fractal is self-similar) and P is 4 (each segment decomposes into 4 smaller segments). The dimension of this fractal is $\ln(P)/\ln(M) = \ln(4)/\ln(3) = 1.2619$.

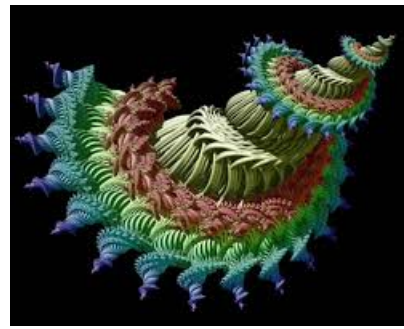
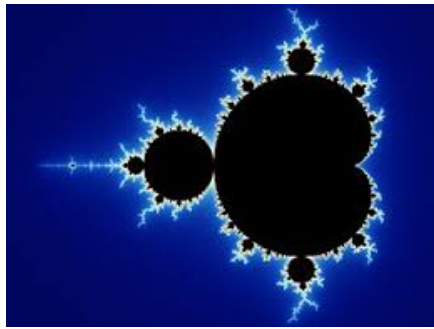
As the iterations continue forever, it can be shown that the length of the Koch curve is **infinitely long**.

The Koch Snowflake:

The Koch Snowflake is generated in very much the same way as is the Koch Curve. Instead of the initiator being a line segment divided into three equal segments, it is an equilateral triangle with each side divided into three equal parts. The generator rule for the Koch curve is then applied to each side.

It is interesting to note that the Koch Snowflake has a **finite area** while its **perimeter is infinite**. As with all fractals, it is **self-similar**. No matter how much we zoom on the curve, it always looks the same.

There are many other fractals, the most famous of all being the Mandelbrot fractals named after its famous discoverer.



To create the Mandelbrot set, pick an arbitrary point C on the complex plane and as a generator use an algorithm that uses the recursive formula:

$$Z_n = Z_{n-1}^2 + C$$

where $Z_0 = 0$. The algorithm will typically generate points that lie either inside or outside the Mandelbrot set. For computer generated art, the point inside the set can be assigned one color while those outside the set can be assigned colors that depend on how many iterations have been required to determine that they are outside the set.

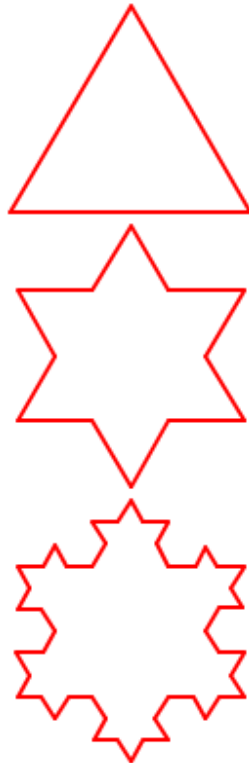
Applications/ Uses of fractals:

- Architecture
- Astrophysics
- Biological sciences
- Computer graphics
- Signal and image compression
- Computer networks
- Medicine
- Neuroscience
- Geography
- Diagnostic imaging

- Natural phenomena
 - Clouds
 - River networks
 - Fault lines
 - Mountain ranges
 - Craters
 - Lightning bolts
 - Coastlines
 - Animal coloration patterns
 - Heart rates
 - Earthquakes
 - Snow flakes
 - Crystals
 - Blood vessels
 - Ocean waves
 - DNA
 - Psychological subjective perception
 - Soil pores

Assignment:

1. Begin with a large equilateral triangle and do two more iterations of the generator rule for a Koch Snowflake. Calculate the dimension of the Koch Snowflake.



2. Begin with an initiator as a vertical line segment. Show two iterations of this “tree” using the generator rule: “Append to the end of the previous segment, two other “branching segments” $\frac{2}{3}$ the length of the previous segment and at angles of 30° from the direction of the original segment.”

